Area Under the ROC Curve of Enhanced Energy Detector

Syed Safwan Khalid and Shafayat Abrar
Department of Electrical Engineering
COMSATS Institute of Information Technology
Chak Shahzad, Islamabad 44000, Pakistan.
Email: {safwan_khalid, sabrar}@comsats.edu.pk

Abstract—The area under the Receiver-Operating-Characteristic (ROC) curve of a detector is a simple and suitable figure-of-merit of its detection capability. However the problem of determining the Area-Under-the-Curve (AUC) of an ROC is a non-trivial exercise and hence it has gone usually unnoticed in literature. In this report we analyze the AUC of the recently proposed enhanced energy detector for spectrum sensing. We derive a simple expression for the AUC which can also be applied to determine the AUC of the conventional energy detector as a special case.

Index Terms—Receiver Operating Characteristics (ROC), Energy Detection, Enhanced Energy Detection, Spectrum Sensing, Area Under the Curve (AUC)

I. INTRODUCTION

Wireless spectrum is a costly resource which is severely under utilized currently, owing to fixed spectrum allocation practices. Cognitive radio (CR) is a promising future technology that could dramatically increase the efficiency of spectrum utilization by dynamically allocating the spectrum to different users. A device equipped with CR technology would essentially perform two major functions: 1) detect spectrum holes in a geographical area i.e. those bands of frequency spectrum which are not being used by any other communication device and 2) configure the communication parameters of the device e.g. transmit power, channel coding and others according to the radio channel characteristics so as to optimize some cost function i.e. throughput, interference etc. The first of these two functions is termed as spectrum sensing and is considered to be the key technology behind a CR device. The problem of spectrum sensing is thus to detect the presence of a signal from a user (termed as primary user) and differentiate it with pure noise i.e. absence of the primary user. Mathematically it is modeled as a binary hypothesis testing problem defined as

\[
y(n) = \begin{cases} 
    u(n), & H_0 \\
    s(n) + u(n), & H_1 
\end{cases} \quad \text{where } n = 1, \ldots, N
\]

(1)

where \( u(n) \) is the white gaussian noise i.e. \( u(n) \sim \mathcal{N}(0, \sigma_u^2) \) and \( s(n) \) is the primary user which is modeled as a zero mean Gaussian random variable with variance \( \sigma_s^2 \). Due to the significance of spectrum sensing in CR a large number of sensing methods have been proposed in literature. Some exploit a-priori information of the transmission characteristics of the primary user e.g. matched filtering technique, spectrum sensing based on cyclostationarity, while others may only depend upon the statistics of the received signal and interference e.g. Energy detection, Improved energy detection etc. A detailed survey of spectrum sensing techniques is discussed in [1].

Typically hypothesis testing is achieved by calculating a decision statistics (lets call it \( v \)) which is compared with a threshold (lets call is \( \lambda \)). If the magnitude of decision statistic exceeds \( \lambda \) then the decision of the detector is \( H_1 \) otherwise the decision is null hypothesis \( H_0 \). The performance of such a detector is expressed in terms of probability of false alarm \( P_F = P(v > \lambda; H_0) \) which quantifies the probability of falsely detecting a primary user when there was none and probability of correct detection or simply probability of detection \( P_D = P(v' > \lambda; H_1) \) which gives the probability of correctly identifying the presence of a primary user. The performance of the detector is completely characterized by its ROC which is a plot of \( P_D \) against \( P_{FA} \) for a given set of parameters e.g. Sample size, SNR etc. As the threshold \( \lambda \) is varied from \(-\infty \rightarrow \infty \) it traverses the entire ROC curve from the upper right point where both probabilities are \((1,1)\) to the lower left point where both are \((0,0)\). Although ROC completely describes the performance of a detector, however, there might be some applications where a single number as a figure-of-merit is required instead of a complete curve. In such a case area under the ROC curve is a suitable candidate. The use of area under the ROC as a suitable figure-of-merit is justified by the Area Theorem [2] and the results derived in [3]. The derivation of a closed-form expression for AUC is a non-trivial exercise for most of the detectors [4] (and references therein). Hence there is little literature available that discuss the analysis of AUC. In [4],[5] Atapattu et al. have derived a closed form expression of AUC for energy detection technique used for spectrum sensing and also discussed the use of complementary AUC when the value of SNR is large. A closed form expression of AUC for energy based detector in relay networks has been recently discussed in [6]. To the best of our knowledge there is no literature available that has discussed the closed form evaluation of the improved/enhanced energy detector proposed in [7],[8]. The rest of the paper is distributed as follows, In Section II, we discuss the energy detector and the enhanced energy detector. In Section III, the closed form expression of AUC for the enhanced energy detector is derived. In Section IV, the numerical and analytical results are compared.
and Section V contains the conclusions.

II. ENHANCED ENERGY DETECTOR

Energy detector is an efficient method for spectrum sensing in CR owing to its very simple structure. Although its accuracy is less than any method that may exploit a priori information of the transmission characteristics of the primary user, however, the reduced complexity associated with energy detector makes it a preferred choice in most scenarios. It does not require any channel state information or any a priori information of the primary user. Its only requirement, apart from the received samples is the information of the noise power at the receiver end to calculate the decision statistic. Energy detector has been proposed and thoroughly discussed in [9], [10]. The decision statistic in (5) follows approximately Gamma distribution [7] with mean and variance given as

\[ \mu_0 := E[v; H_0] = \frac{2^m/2}{\sqrt{\pi}} \Gamma \left( \frac{m+1}{2} \right) \]  
\[ \mu_1 := E[v; H_1] = \frac{2^m/2(1+\gamma)^{m/2}}{\sqrt{\pi}} \Gamma \left( \frac{m+1}{2} \right) \]  
\[ \sigma_0 := \text{var}[v; H_0] = \frac{1}{N} \frac{2^m}{\sqrt{\pi}} \left[ \Gamma \left( \frac{2m+1}{2} \right) - \frac{1}{\sqrt{\pi}} \Gamma^2 \left( \frac{m+1}{2} \right) \right] \]  
\[ \sigma_1 := \text{var}[v; H_1] = \frac{1}{N} \frac{2^m(1+\gamma)^m}{\sqrt{\pi}} \left[ \Gamma \left( \frac{2m+1}{2} \right) - \frac{1}{\sqrt{\pi}} \Gamma^2 \left( \frac{m+1}{2} \right) \right] \]  

where \( \gamma = \sigma_0^2/\sigma_1^2 \) is the signal-to-noise ratio. For the enhanced energy detector the probability of false alarm is given as

\[ P_f = P(v > \lambda; H_0) = 1 - F_{\nu,H_0}(\lambda, k_0, \theta_0) \]  
\[ P_d = P(v > \lambda; H_1) = 1 - F_{\nu,H_1}(\lambda, k_1, \theta_1) \]

where \( F_{\nu,H}(\cdot; \cdot; \cdot) \) is the cumulative distribution function (CDF) of the decision statistic under hypothesis \( H_0 \), \( k_0 = \mu_0^2/\sigma_0^2 \) is the shape parameter of the Gamma distributed \( v \) and \( \theta_0 = \sigma_0/\mu_0 \) is the scale parameter. For a given \( P_f \) the probability of detection is given as

It is readily observed that the expressions in (10) and (11) are intractable, hence [7] does not provide any analytical results to determine the optimum value of \( m \), instead it relies on numerically computed curves or tables from which the optimum value of \( m \) can be determined for a given \( P_f \), \( P_d \), \( N \) and SNR. In [8] Song et al. have also proposed the same detector with arbitrary power \( m \) instead of squaring, they have termed it as Enhanced energy detector. The difference in their approach is that instead of approximating (5) as Gamma distributed they have used the Gaussian approximation which simplifies the analysis. There is also given some analysis to find the optimum value of \( m \) however the analysis does not yield a closed-form formula. Under the Gaussian approximation the expression of \( P_f \) and \( P_d \) are given as follows

\[ P_f = Q \left( \frac{\lambda - \mu_0}{\sqrt{\frac{\sigma_0^2}{N}}} \right) \]  
\[ P_d = Q \left( \frac{\lambda - \mu_1}{\sqrt{\frac{\sigma_1^2}{N}}} \right) \]

where the function \( Q(x) \) is defined as:

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp \left( -\frac{t^2}{2} \right) \, dt \]

The value of SNR has a large influence on the approximation of Gamma distribution as Gaussian distribution and the approximation is only valid for low values of SNR [8]. However, it is not a serious limitation since in practice, spectrum sensing is usually performed at very low SNR values. Moreover we have observed that the approximation error increase for very small or very large values of \( m \). In Fig.1, we have plotted a number of ROC curves for different values of SNR to show the validity of the Gaussian approximation. It is seen that the approximation error slightly increases as SNR increases however for low values of SNR it may safely be used instead of Gamma approximation.

\[ v = \frac{1}{N \sigma_u^2} \sum_{n=0}^{N-1} |y(n)|^2 \]  

where \( \Gamma(\cdot; \cdot) \) is the incomplete gamma function and \( Q_u(\sqrt{2\gamma}, \sqrt{\lambda}) \) is the Marcum-Q function.

In [7] Chen reasoned that although the energy detector maximizes the generalized likelihood function however it may not maximize the \( P_d \) for a given set of \( P_f \), \( N \) and SNR similarly it may not minimize \( P_f \) for a given set of \( P_d \), \( N \) and SNR. To add another degree of freedom there was introduced an arbitrary power \( m \) instead of the squaring of magnitude used in the energy detector.
In the next section we use (12) and (13) to determine the area under the ROC curve for the improved/enhanced energy detector.

III. DERIVATION OF AUC

The area covered by the ROC curve of $P_d$ vs. $P_f$ is given as follows

$$A(\gamma) = \int_{0}^{1} P_d(\gamma, \lambda) dP_f(\lambda)$$

(14)

where $\lambda$ is the threshold parameter and $\gamma$ is the value of SNR. Note that, the above equation can also be written as an integration with respect to $\lambda$:

$$A(\gamma) = -\int_{-\infty}^{\infty} P_d(\gamma, \lambda) \frac{\partial P_f(\lambda)}{\partial \lambda} d\lambda$$

(15)

The partial derivative $\frac{\partial P_f(\lambda)}{\partial \lambda}$ is evaluated as follows:

$$\frac{\partial P_f}{\partial \gamma} = \frac{\partial}{\partial \gamma} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{t^2}{2}\right)}{\sqrt{\sigma_0^2/\pi}} dt$$

(16)

Introducing a change of variable $x = \frac{\sqrt{2\pi}t}{\sigma_0} + \mu_0$, Eq. (16) becomes

$$\frac{\partial P_f}{\partial \gamma} = \frac{N}{\sqrt{2\pi} \sigma_0^2} \times$$

$$\left[\frac{\partial}{\partial \gamma} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(x - \mu_0\right)\frac{N}{\sigma_0^2}\right)^2 dx\right]$$

(17)

Now putting (17) in (15).

$$A(\gamma) = \frac{N}{\sqrt{2\pi} \sigma_0^2} \int_{-\infty}^{\infty} \frac{Q\left(\frac{\gamma - \mu_1}{\sigma_0^2}\right)}{\sqrt{2\pi}} d\gamma$$

(18)

Again introducing a change of variable $\gamma' = (\gamma - \mu_0)\sqrt{\sigma^2/\sigma_0^2}$ and let $a = \frac{\mu_0 - \mu_1}{\sqrt{\sigma^2/\sigma_0^2}}$ and $b = \sqrt{\sigma_0^2/\sigma_1^2}$, we get

$$A(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(\gamma'b + a) \exp\left(-\frac{\gamma'^2}{2}\right) d\gamma'$$

(19)

The integral in (17) is evaluated as follows (please refer to Appendix):

$$A(\gamma) = Q\left(\frac{N(\mu_0 - \mu_1)^2}{\sigma_0^2 + \sigma_1^2}\right)$$

(20)

Note that Eq. (20) gives a simple expression of AUC for the enhanced energy detector as a function of SNR.

IV. NUMERICAL RESULTS

We have calculated the values of AUC using the analytical expression derived in (20) and compared them against the values calculated by directly computing (14) using numerical integration techniques. The $P_{FA}$ and $P_D$ used in (14) are calculated using the expressions in (10), (11) (i.e. Gamma approximation) and also using (12) and (13) (i.e. Gaussian approximation). In Fig. 2 the values of AUC, for different values of sample size $N$ and SNR are shown against a range of values of $m$. It can be seen that the analytical results are in complete accordance with the numerically computed results. Moreover it is observed that for the values of $N$ and SNR the conventional energy detector is optimal from the standpoint of maximizing AUC. In Fig. 3 the values of AUC are plotted against a range of SNR, for some values of $m$ and $N$. Again we see that the analytical results match the results obtained numerically. We observe a slight deviation in case of $m = 4$ for values of SNR close to 0. This deviation is owing to the fact (as explained earlier) that the error of approximating Gamma distribution with Normal distribution increases with increasing values of SNR and $m$. In TABLE.1 we have tabulated the results of our analytical expression employed specifically for the conventional energy detector by setting the value of $m = 2$. The values of AUC calculated by computing (14) using numerical integration techniques are also tabulated for comparison, where $P_{FA}$ and $P_D$ are used from (3) and (4). It is observed that the analytical closed form expression proposed in this paper closely matches the numerical results for conventional energy detector.
however our proposed closed form expression using Gaussian approximation is much simpler than the exact formula derived in [4] hence it is more suitable for real time applications that might require the AUC measurements of the energy detector. It is also observed that for low values of SNR and large sample size, the conventional energy detector maximizes the area under the ROC curve.

APPENDIX

DERIVATION OF (20)

From (19) we have
\[ A(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q (a + \gamma b) \exp \left( -\frac{\gamma^2}{2} \right) d\gamma' \] (21)

Using the definition of \( Q \) function it can be written as
\[ A(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{t'^2}{2} - \frac{\gamma'^2}{2} \right) dt d\gamma' \] (22)

Let \( \theta \) be the angle between the line \( t = a + b\gamma' \) and x-axis then introducing a rotation of co-ordinates using the transformations
\[ \gamma' = \gamma'' \cos \theta - t' \sin \theta \]
\[ t = \gamma'' \sin \theta + t' \cos \theta \] (23)

The integral in (22) can be written as
\[ I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{t'^2}{2} - \frac{\gamma'^2}{2} \right) dt' = Q \left( \frac{a}{\sqrt{b^2 + 1}} \right) \] (24)

(For a graphical interpretation and detailed derivation of (24) refer to [12],Appendix C). Inserting the values of \( a \) and \( b \) in (24) we get (20).

REFERENCES


V. CONCLUSIONS

AUC is a simple and effective figure of merit of the detection capabilities of a hypothesis testing algorithm. A simple analytical expression of AUC for the enhanced energy detector for spectrum sensing has been derived using Gaussian approximation of the Gamma distributed decision statistic. The results show that for low SNR values the analytical results match the results computed using numerical techniques. The analytical closed-form expression obtained can also be applied to the conventional energy detector as a special case. It should be noted that although the problem of area under the ROC for conventional energy detector has been discussed at length in [4] and

![Fig. 2. Area under the ROC curve plotted against different values of m.](image2)

![Fig. 3. Area under the ROC curve plotted against different values of SNR.](image3)

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<th>SNR(dB)</th>
<th>AUC (Numerical)</th>
<th>AUC (Analytical)</th>
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TABLE I COMPARISON OF AUC VALUES FOR THE TRADITIONAL ENERGY DETECTOR COMPUTED NUMERICALLY AGAINST THE VALUES GIVEN BY THE DERIVED ANALYTICAL EXPRESSION BY SETTING \( m = 2 \) AND \( N = 100 \).