Lower Partial Moments-Proxy of Downside Risk

Usman Ayub (Corresponding Author)
Assistant Professor and PhD Scholar,
COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Dr. Qaiser Abbas
Professor, Department of Management Sciences,
COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Syed Kashif Saeed
PhD Scholar, COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Shahid Mehmmod Sargana
Assistant Professor, COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Abstract

CAPM with all its controversies is still valid, popular and most used asset pricing model in world. Three major perspectives run among the researchers. The first rejects CAPM completely and the rest two advocates changes but differently. A scrupulous analysis yields that investors care more for downside risk rather than variance. Overwhelming evidence especially in emerging markets is present world-wide. Stochastic Dominance is a powerful tool to measure downside risk. Lower Partial Moment is developed proxy for Stochastic Dominance. Lower Partial Moment working comes under scrutiny. In last, its algorithm and relation of lower partial moment to downside-beta is discussed.

Key Words: CAPM, Downside risk, Lower partial moment.

I. Introduction

This paper discusses the primary issues of CAPM and finds controversies world-wide around CAPM. The major among them are investor’s preference for downside risk and non-normality in asset returns. This paper, being its own kind, brings the world of CAPM and downside risk together with lower partial moment as a bridge between them. Lower partial moment is a proxy for downside risk. The paper theme lower partial moment and for the first time brings together all the relevant material related to it with discussing in detail. Lower partial moment algorithms are discussed and loopholes are also identified making it very easier to be adopted. In the end, the relationship between lower partial moment and CAPM, and downside beta is established. The paper also claims to strike the balance by addressing the theoretical issues of lower partial moment, making it simpler to comprehend for beginners in financial econometrics and advance.
The paper is divided into four sections. The first section gives an introduction while second section presents a brief introduction of CAPM. The third section discusses the problems with CAPM and the evidence in favor of downside beta. Lower partial moment is discussed theoretically. Initially, lower partial moment is ascertained as a proxy of downside risk and then its usage in research is discussed in detail. The last section is conclusion with recommendations.

2. Introduction to CAPM

Capital Asset pricing Model (CAPM) has mystified the researchers in the field of finance and economics for the last half a century. Despite the theoretical and empirical onslaught on it, it has survived and on the contrary has gained more popularity and acceptability both among researchers as well as players in stock markets. It is simple, cost-effective and even a beginner in the field of finance easily comprehends it.

Markowitz and Tobin’s contributions to the field of asset pricing have made CAPM its birth-child. With almost all the conditions of CAPM having roots in MV-framework and 2-fund separation theorem, CAPM cannot be independently analyzed. The pillars of CAPM are Capital Market Line (CML) and Security Market Line (SML). CML portrays a linear relationship of risk and return among different portfolios. And its test ground is SML where the performers of financial econometrics show their metal.

With evidence against CAPM from developed markets and mix evidence from emerging markets, it seems that theoretical and empirical framework of CAPM is to be reconsidered, rethought and regenerated. Researchers have paid attention to this and have come up with a solution. They contest that investors do really care about negative returns, do not give equal weights to upside and downside risk and are safety-conscious towards a below target return and they call it the “downside risk”. In life, we do not find what we dream but we tend to come as closer to it. In finance we call it proxy. Lower partial moment based on stochastic dominance, is considered a proxy of downside risk. Though developed in seventies and did not gain due attention until recently when researchers retraced their footsteps back in time to correct the assumptions of investor’s preference of risk and the condition of ‘normality” for CAPM.

3. The Background

During seventies, earlier tests for CAPM indicate that favorability but later on it seems that research on CAPM and controversies around it are highly correlated. These controversies revolve around three major viewpoints. Roll (1977) and Ross (1977) are the forerunners in the first perspective which completely rejects CAPM theoretically as well as empirically. Shanken (1982, 1987), Kandel and Stambaugh (1987), Reisman (1992) and Nawalkha (1997,}

---

22 CAPM is independently developed by Sharpe (1964), Lintner (1965), Mossin (1966) & Treynor(1962).
23 73.5% of U.S. CFOs (Graham & Harvey (2001)) and 45% in Europe use CAPM (Brounen, Abe de Jong and Koedijk (2004)).
24 Mean-Variance (MV) Approach (Markowitz (1952)).
25 Two-Fund Separation theorem under the conditions of market equilibrium (Tobin (1958)).
26 See Beaver, Ketter and Scholes (1970), Galai and Masulis (1976), Black, Jensen and Scholes (1972), Fama and MacBeth (1973) and Blume and Friend (1973).
27 Ross (1976) applies Arbitrage Pricing Theory (APT) to capital asset pricing and contests that APT substitutes CAPM.

The third perspective is kind of solution provider. In it, the theoretical shortcomings of CAPM are addressed and transformed into modified versions of CAPM. Fischer Black (1972) provides solution to the problem without risk-free borrowing or lending in CAPM and is supported by Stambaugh (1982). Beta’s instability over time is disentangled by Bollerslev (1986, 1988, 1994) and French, Schwert, and Stambaugh (1987) and Jagannathan and Wang (1996) and held up by Lettau and Ludvigson (2001), Zhang (2003) and Petkova and Zhang (2003).

The million dollar question arises as to discard CAPM or not! A meticulous analysis offers a solution. If we go back in time to the very basics as to how an investor defines and assigns value to risk perhaps we can find a solution. CAPM basics imply that an investor cares for risk in Markowitz’s (1952) Mean-Variance (MV) framework. However there are two major problems associated to it. Firstly investors care more for downside risk as compared to upside risk. And secondly, to validate CAPM “normality” is assumed which is very much questionable. The irony is that you propose something and believe something works better. This is what happened to Markowitz himself. Initially, he is convinced that downside risk as semivariance is more appropriate as compared to variance as a measure of investor’s risk. However he stays with the variance but does recognize the importance of semivariance\(^ {29}\) (Markowitz (1959)).

### 3.1 Are Downside Risk and Normality Really Fundamental

Challenging the theoretical MV-framework of CAPM is altogether not simple\(^ {30}\). The first major premise is that investors are safety cognizant and the second one is the empirical tests are based on condition on normal distribution. However the latter problem can be addressed by using Generalized Method of Moments (GMM) as normal distribution is a sufficient assumption which is assumed for statistical purposes and to make it empirically testable (Campbell, Lo and MacKinlay (1997), chapter 5). But still this condition does affect the empirical results and

---


\(^{29}\) He discards it due to lack of resources at that time. Variance is computationally simpler as semivariance optimization model requires twice the number of data inputs than the variance model and with the lack of cost-effective computer power until advent of the microcomputer in 1980s and with the fact that variance model is already mathematically very complex.

\(^{30}\) For detail, please see Abbas (2011)
though one can make CAPM testable but one cannot forget message it is conveying as to the investor’s preference for the downside utility function.

Downside risk is initially discarded by Markowitz (1952, 1959) and lived for three decades under the shadow of CAPM but it slowly makes its way to the arena. Roy (1952) starts the debate of downside risk and argues that investor care for disaster and safety from that disaster is the foremost goal of an investor thus entailing safety-first rule (SF-rule). Taken true means that Markowitz MV-framework and Roy’s SF-rule do not have any common ground to hold except under the condition of “normality” which is hard to find (Fama (1965a&b), Kon (1984), Affleck-Graves and McDonald (1989), Richardson and Smith (1993), Susmel (2001), Hwang and Pedersen (2002), Dufour, Khalaf and Beaulieu (2003)). In addition, Aparicio and Estrada (1997) study Scandinavian markets and assert that fat tails and high peaks is shown by distributions of daily stock returns. Bekaert and Harvey (1995, 1997), Eftekhari and Satchell (1996) and Bekaert, Erb, Harvey, and Viskanta (1998), show that emerging market display non-normality. Researchers follow a pragmatic approach and prefer normal distribution being not a necessary condition to derive CAPM primarily assumed for statistical purposes. However under large departure from normality particularly when the distribution is severely asymmetric, MV-criterion may fail to correctly approximate the expected utility (see Chunachinda et al. (1997), Athayde and Flôres (2004) and Jondeau and Rockinger (2006)). A large body of literature is available purposing that MV-framework lacks theoretical support and stresses higher moments (skewness and kurtosis) are to be included.


Investors are indifferent to upside and downside risk is not true. They do not give equal weights to both upside and downside risk as assumed in CAPM. Markowitz (1959) himself agrees to it and Kahneman and Tversky (1979), Gul (1991) and Estrada (2000, 2002 and 2007) give reinforcement. Post and Levy (2005) emphasize that if investors show different behavior for bear and bull markets, then they are willing to pay premium for stocks giving downside protection in bear markets and upside potential in bull markets. Under this assumption one wonders how still CAPM survives the blitz directed to it all these decades with variance as the measure of risk.

31 Both Nielsen (1990) and Allingham (1991) provide sufficient conditions but these conditions are hard to interpret. Also see Berk (1997).
32 CAPM rests on MV-framework.
33 Downside risk is different from co-skewness risk. The downside risk is captured non-linearly while co-skewness does not assume it.
34 Especially for skewness and kurtosis.
35 Investors are risk averse for losses and they become seekers for gains.
Lower Partial Moments-Proxy of Downside Risk

Usman Ayub (Corresponding Author)
Assistant Professor and PhD Scholar,
COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Dr. Qaiser Abbas
Professor, Department of Management Sciences,
COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Syed Kashif Saeed
PhD Scholar, COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Shahid Mehmmod Sargana
Assistant Professor, COMSATS Institute of Information Technology,
Chak Shahzad, Park Road, Islamabad.

Abstract

CAPM with all its controversies is still valid, popular and most used asset pricing model in world. Three major perspectives run among the researchers. The first rejects CAPM completely and the rest two advocates changes but differently. A scrupulous analysis yields that investors care more for downside risk rather than variance. Overwhelming evidence especially in emerging markets is present world-wide. Stochastic Dominance is a powerful tool to measure downside risk. Lower Partial Moment is developed proxy for Stochastic Dominance. Lower Partial Moment working comes under scrutiny. In last, its algorithm and relation of lower partial moment to downside-beta is discussed.

Key Words: CAPM, Downside risk, Lower partial moment.

I. Introduction

This paper discusses the primary issues of CAPM and finds controversies world-wide around CAPM. The major among them are investor’s preference for downside risk and non-normality in asset returns. This paper, being its own kind, brings the world of CAPM and downside risk together with lower partial moment as a bridge between them. Lower partial moment is a proxy for downside risk. The paper theme lower partial moment and for the first time brings together all the relevant material related to it with discussing in detail. Lower partial moment algorithms are discussed and loopholes are also identified making it very easier to be adopted. In the end, the relationship between lower partial moment and CAPM, and downside beta is established. The paper also claims to strike the balance by addressing the theoretical issues of lower partial moment, making it simpler to comprehend for beginners in financial econometrics and advance

21 The investigation in CAPM have lead so far to three working papers and This paper is a direct extension of the first paper, which are written while working for PhD dissertation at COMSATS Institute of Information Technology, Islamabad.
portfolio analysis and investment. The paper also claims that CAPM is not rejected altogether but modified into downside risk with lower partial moment being downside risk’s heir.

The paper is divided into four sections. The first section gives an introduction while second section presents a brief introduction of CAPM. The third section discusses the problems with CAPM and the evidence in favor of downside beta. Lower partial moment is discussed theoretically. Initially, lower partial moment is ascertained as a proxy of downside risk and then its usage in research is discussed in detail. The last section is conclusion with recommendations.

2. Introduction to CAPM
Capital Asset pricing Model (CAPM)\(^2\) has mystified the researchers in the field of finance and economics for the last half a century. Despite the theoretical and empirical onslaught on it, it has survived and on the contrary has gained more popularity and acceptability\(^3\) both among researchers as well as players in stock markets. It is simple, cost-effective and even a beginner in the field of finance easily comprehends it.

Markowitz\(^4\) and Tobin’s\(^5\) contributions to the field of asset pricing have made CAPM its birth-child. With almost all the conditions of CAPM having roots in MV-framework and 2-fund separation theorem, CAPM cannot be independently analyzed. The pillars of CAPM are Capital Market Line (CML) and Security Market Line (SML). CML portrays a linear relationship of risk and return among different portfolios. And its test ground is SML where the performers of financial econometrics show their metal.

With evidence against CAPM from developed markets and mix evidence from emerging markets, it seems that theoretical and empirical framework of CAPM is to be reconsidered, rethought and regenerated. Researchers have paid attention to this and have come up with a solution. They contest that investors do really care about negative returns, do not give equal weights to upside and downside risk and are safety-conscious towards a below target return and they call it the “downside risk”. In life, we do not find what we dream but we tend to come as closer to it. In finance we call it proxy. Lower partial moment based on stochastic dominance, is considered a proxy of downside risk. Though developed in seventies and did not gain due attention until recently when researchers retraced their footsteps back in time to correct the assumptions of investor’s preference of risk and the condition of ‘normality” for CAPM.

3. The Background
During seventies, earlier tests for CAPM indicate that favorability\(^6\) but later on it seems that research on CAPM and controversies around it are highly correlated. These controversies revolve around three major viewpoints. Roll (1977) and Ross (1977) are the forerunners in the first perspective which completely rejects CAPM theoretically as well as empirically\(^7\). Shanken (1982, 1987), Kandel and Stambaugh (1987), Reisman (1992) and Nawalkha (1997, 2000).

---

\(^{22}\) CAPM is independently developed by Sharpe (1964), Lintner (1965), Mossin (1966) & Treynor22(1962).

\(^{23}\) 73.5% of U.S. CFOs (Graham & Harvey (2001)) and 45% in Europe use CAPM (Brounen, Abe de Jong and Koedijk (2004)).

\(^{24}\) Mean-Variance (MV) Approach (Markowitz (1952)).

\(^{25}\) Two-Fund Separation theorem under the conditions of market equilibrium (Tobin (1958)).

\(^{26}\) See Beaver, Kettler and Scholes (1970), Galai and Masulis (1976), Black, Jensen and Scholes (1972), Fama and MacBeth (1973) and Blume and Friend (1973).

\(^{27}\) Ross (1976) applies Arbitrage Pricing Theory (APT) to capital asset pricing and contests that APT substitutes CAPM.

The third perspective is kind of solution provider. In it, the theoretical shortcomings of CAPM are addressed and transformed into modified versions of CAPM. Fischer Black (1972) provides solution to the problem without risk-free borrowing or lending in CAPM and is supported by Stambaugh (1982). Beta’s instability over time is disentangled by Bollerslev (1986, 1988, 1994) and French, Schwert, and Stambaugh (1987) and Jagannathan and Wang (1996) and held up by Lettau and Ludvigson (2001), Zhang (2003) and Petkova and Zhang (2003).

The million dollar question arises as to discard CAPM or not! A meticulous analysis offers a solution. If we go back in time to the very basics as to how an investor defines and assigns value to risk perhaps we can find a solution. CAPM basics imply that an investor care for risk in Markowitz’s (1952) Mean-Variance (MV) framework. However there are two major problems associated to it. Firstly investors care more for downside risk as compared to upside risk. And secondly, to validate CAPM “normality” is assumed which is very much questionable. The irony is that you propose something and belief something works better. This is what happened to Markowitz himself. Initially, he is convinced that downside risk as semivariance is more appropriate as compared to variance as a measure of investor’s risk. However he stays with the variance but does recognize the importance of semivariance29 (Markowitz (1959)).

3.1 Are Downside Risk and Normality Really Fundamental

Challenging the theoretical MV-framework of CAPM is altogether not simple30. The first major premise is that investors are safety cognizant and the second one is the empirical tests are based on condition on “normal distribution. However the latter problem can be addressed by using Generalized Method of Moments (GMM) as normal distribution is a sufficient assumption which is assumed for statistical purposes and to make it empirically testable (Campbell, Lo and MacKinlay (1997), chapter 5). But still this condition does affect the empirical results and

---


29 He discards it due to lack of resources at that time. Variance is computationally simpler as semivariance optimization model requires twice the number of data inputs than the variance model and with the lack of cost-effective computer power until advent of the microcomputer in 1980s and with the fact that variance model is already mathematically very complex.

30 For detail, please see Abbas (2011)
though one can make CAPM testable but one cannot forget message it is conveying as to the investor’s preference for the downside utility function.

Downside risk is initially discarded by Markowitz (1952, 1959) and lived for three decades under the shadow of CAPM but it slowly makes its way to the arena. Roy (1952) starts the debate of downside risk and argues that investor care for disaster and safety from that disaster is the foremost goal of an investor thus entailing safety-first rule (SF-rule). Taken true means that Markowitz MV-framework and Roy’s SF-rule do not have any common ground to hold except under the condition of “normality” which is hard to find (Fama (1965a&b), Kon (1984), Affleck-Graves and McDonald (1989), Richardson and Smith (1993), Susmel (2001), Hwang and Pedersen (2002), Dufour, Khalaf and Beaulieu (2003)). In addition, Aparicio and Estrada (1997) study Scandinavian markets and assert that fat tails and high peaks is shown by distributions of daily stock returns. Bekaert and Harvey (1995, 1997), Eftekhari and Satchell (1996) and Bekaert, Erb, Harvey, and Viskanta (1998), show that emerging market display non-normality.

Researchers follow a pragmatic approach and prefer normal distribution being not a necessary condition to derive CAPM primarily assumed for statistical purposes. However under large departure from normality particularly when the distribution is severely asymmetric, MV-criterion may fail to correctly approximate the expected utility (see Chunachinda et al., Athayde and Flôres (2004) and Jondeau and Rockinger (2006)). A large body of literature is available purposing that MV-framework lacks theoretical support and stresses higher moments (skewness and kurtosis) are to be included.


Investors are indifferent to upside and downside risk is not true. They do not give equal weights to both upside and downside risk as assumed in CAPM. Markowitz (1959) himself agrees to it and Kahneman and Tversky (1979), Gul (1991) and Estrada (2000, 2002 and 2007) give reinforcement. Post and Levy (2005) emphasize that if investors show different behavior for bear and bull markets, then they are willing to pay premium for stocks giving downside protection in bear markets and upside potential in bull markets. Under this assumption one wonders how still CAPM survives the blitz directed to it all these decades with variance as the measure of risk.

---

31 Both Nielsen (1990) and Allingham (1991) provide sufficient conditions but these conditions are hard to interpret. Also see Berk (1997).
32 CAPM rests on MV-framework.
33 Downside risk is different from co-skewness risk. The downside risk is captured non-linearly while co-skewness does not assume it.
34 Especially for skewness and kurtosis.
35 Investors are risk averse for losses and they become seekers for gains.
Quirk and Saposnik (1962), Mao (1970), Klemkosky (1973) and Ang and Chua (1979) all demonstrate that semivariance is superior to variance. Grootveld and Hallerbach (1999) conclude that downside risk approaches tend to favor more as compared to variance. Harvey (2000) suggests the supremacy of downside risk for emerging markets for a sample based on equilibrium as well as non-equilibrium-based risk measures. Estrada (2002) uses methodology of Levy and Markowitz (1979), asserts credibility of semivariance over standard deviation-empirically. Balzer (2001) comes up with the relationship between utility and downside risk measures, and contests that semivariance framework is better than Markowitz’s M-V framework. Ang, Chen and Xing (2002, 2006) conclude that downside beta works better in predicting future than variance-based framework. Estrada (2002) affirms that over 45% of variability for a cross-sectional analysis for a joint sample of developed markets and emerging markets and 55% in emerging market are explained by DCAPM. Post and Vliet (2004) contest that DCAPM surpasses CAPM. Estrada and Serra (2005) find out global downside beta is the best risk measures among the various risk measures studied. This evidence is just the beginning, ahead lays vast frontiers of downside risk in form of its conditional and intertemporal versions. The basic premise of this debate is to establish downside risk the heir of CAPM in field of asset pricing but the question arises as to how downside risk is measured in asset returns.

3.2: Lower Partial Moment-Proxy for Downside Risk

Researchers though advocate supremacy of downside risk over variance as a measure of investors’ risk but face an uphill task of an appropriate proxy for downside risk in asset returns. Markowitz (1952, 1959) and Roy (1952) stress semivariance but the question arises how to measure downside risk! Markowitz has two suggestions for measuring downside risk: SVm-semivariance from mean return or below-mean semivariance and SVt- semivariance from a target return or below-target semivariance. Markowitz names these measures partial or semivariances. In addition, Hogan and Warren (1974) substitute variance with semivariance but it fails to get due attention until the adoption of stochastic dominance in downside risk.

Stochastic dominance (SD) is a powerful risk analysis tool which does not rely on any certain type of distribution but, rather compares results in pairs using their cumulative distribution functions. In portfolio science, an investor exhibit three type of behavior namely; investor prefers more to less (nonsatiation), investor is risk averse and prefer more wealth but at a decreasing rate (decreasing absolute risk aversion (DARA)). Quadratic utility is assumed for variance in MV-framework using Taylor series to expand the utility function including the first two moments; mean and variance and ignoring the rest. However quadratic concave utility curve turns down after they reach the “bliss point”-certainly against the third assumption of the type of investor i.e. DARA. Under these circumstances SD seems to be the ideal tool as it does not...
assumes a specific form of distribution and the different form of SD capture all the three behaviors of an investor\(^{41}\).

SD uses probability distribution functions (pdf) of assets in pairs and asserts the dominance of one over the other. For the first condition SD forms First-order SD (FSD) such as first difference of a utility function is greater than zero i.e. \(U'(w) > 0\). For the first two conditions SD assumes Second-order SD (SSD) as second difference be less than zero i.e. \(U''(w) < 0\) and to fulfill all the three conditions SD assumes Third-order SD (TSD) as the third difference is positive i.e. \(U'''(w) > 0\)\(^{42}\).

The procedure seems simple and can be used both in portfolio optimization as well as for downside risk in asset pricing but the problem to quantify SD remains until the breakthrough by Bawa\(^{43}\).

In 1975, Bawa develops proxy for SD. He generalizes semi-variance measure of risk with the development of the Lower Partial Moment (LPM), proxy for stochastic dominance, to reflect less restrictive class of decreasing absolute risk aversion (DARA) utility function. Bawa uses n-order LPMs to cover a wide range of risk measures as:

\[
LPM_n(\tau, R_i) = \int_{-\infty}^{\tau} (\tau - R_i)^n dF(R_i) ........................................(1)
\]

where \(\tau\) is the target return specified by the investor, \(R_i\) is the return of asset \(i\), \(dF(R_i)\) is probability density function of asset \(i\) and \(n\) is the order of investor’s risk preference for ex ante. For setting LPM=0 (i.e. \(n=0\)) means that the investor is risk neutral. For \(n=1\) means that the investor is risk averse but in a mean-variance framework. For \(n=2\) implies that the investor is risk averse but from the point of semivariance. For \(n=3\) and \(n=4\), represent skewness and kurtosis. For discrete form the equation (1) takes the following form:

\[
LPM_n(\tau, R_i) = \frac{1}{\tau} \sum_{i=1}^{\tau} \left[\text{Max}(0, (\tau, R_i))\right]^n ........................................(2)
\]

Fishburn (1977) extends the general LPM model into unlimited view of LPM which embraces all classes of investors; risk averse, risk seeking and risk neutral. For value \(n<1\) captures risk seeking behavior while setting equal to zero is risk neutral behavior and risk averse behavior is \(n > 1\). The higher the \(n\) (but above one), the higher the degree of risk aversion for an investor\(^{44}\).

Moreover with fractional degrees like 2.33 or 3.89 for different \(n\) in Fishburn classes, LPM simply encompasses unlimited view of risk measures (Nawrocki (1999)).

Bawa and Lindenberg (1977) generalize LPM based on co-semivariance measure of co-LPM or GCLPM for n-degree LPM structures defined as:

\[
GCLPM_{ij,n}(\tau, R_i, R_j) = \int_{-\infty}^{\tau} \int_{-\infty}^{\tau} (\tau - R_i)^{n-1}(\tau - R_j)dF(R_i, R_j)....(3)
\]

where normally the covariance between security \(i\) is not equal to \(j\) as depicted in Markowitz’s EV-framework and \(dF(R_i, R_j)\) is the joint probability density function of returns of asset \(i\) and \(j\). However if one wishes to equal them then it has the flexibility to do so. In this case it will not be asymmetric co-LPM but symmetric co-LPM as follows:

\[
GCLPM_{ij,n}(\tau, R_i, R_j) \neq GCLPM_{ji,n}(\tau, R_j, R_i)..............................(4)
\]

\(^{41}\) Quadratic utility also assumes IARA and IRRA while evidence is in favor of CRRA (Blume and Friend (1973)) and DARA (Cohn et al. (1975)). For detail, see Chavas (2004), chapter 4.

\(^{42}\) For details see Levy (2006).

\(^{43}\) For details see Elton, Gruber, Brown and Goetzmann (2003).

\(^{44}\) For \(n \geq 0\) is FSD, \(n \geq 1\) is SSD and \(n \geq 2\) is TSD-efficient set. For details see Fishburn (1977) and Nawrocki (1991, 1999).
for asymmetric co-LPM,
\[ GCLPM_{ji,n}(\tau, R_i, R_j) = GCLPM_{nj,n}(\tau, R_j, R_i) \] ........................................(5)

for symmetric co-LPM, in this the equation(5) will reduce to:
\[ GCLPM_{ji,n}(\tau, R_i, R_j) = LPM_{mnt}(\tau, R_i) \] ........................................(6)

when \( R_i = R_j \).

For ex post or discrete form, equation (4) takes the following form:
\[ GCLPM_{ji,n}(\tau, R_i, R_j) = \frac{1}{t} \sum_{i=1}^{t} [\text{Max}(0, (\tau, R_i))]^{n-1}(\tau - R_i) \] ..........(7)

In all of above cases, Bawa and Lindenberg (1977) propose that \( \tau \) is equal to risk-free interest rate. However later versions became more flexible and \( \tau \) can assume any value according to investor’s choice45.

3.3 Algorithms for Lower Partial Moment

In a Markowitz world, quadratic programming is used to optimize a portfolio with linear constraints46. As the risk of portfolio is defined as:
\[ \sigma_p^2 = \sum_{i=1}^{n} W_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} W_i W_j \sigma_{ij} \] .................................................(8)

where \( \sigma_p^2 \) is the variance of portfolio, \( W_i \) and \( W_j \) are weights of individual securities \( i \) and \( j \) and \( \sigma_{ij} \) is the covariance between securities \( i \) and \( j \). However it is assumed that under efficient diversification all risk is diversified but the covariance risk cannot be diversified (Sharpe (1964) and Elton, Gruber, Brown and Goetzmann (2003)) so all attention is paid to minimize the following expression:
\[ \text{Minimize } F(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \] .................................................(9)

Subject to the following constraints:
\[ C1: R_p = \sum_{i=1}^{n} x_i (\bar{R}_i - R_p) \]
\[ C2: \sum_{i=1}^{n} x_i = 1 \]
\[ C3: x_i \geq 0, \ i = 1, 2, \ldots, N. \]

where \( x_i \) is portion of portfolio in security \( i \), \( R_p \) is portfolio return and \( R_i \) is return of security. \( C1 \) entails that the portfolio return is equivalent to weighted returns of individual securities. \( C2 \) implies that all weights should equal to 1 and \( C3 \) says that all weights should be more that 0 for no short sales allowed. However if short sales are allowed then \( C3 \) does not stand.

For downside risk maximization, two approaches can be used; Harlow (1991) applies (9) on LPM using the same constraints as discussed above:

46 For details see Elton, Gruber, Brown and Goetzmann (2003).

The third perspective is kind of solution provider. In it, the theoretical shortcomings of CAPM are addressed and transformed into modified versions of CAPM. Fischer Black (1972) provides solution to the problem without risk-free borrowing or lending in CAPM and is supported by Stambaugh (1982). Beta’s instability over time is disentangled by Bollerslev (1986, 1988, 1994) and French, Schwert, and Stambaugh (1987) and Jagannathan and Wang (1996) and held up by Lettau and Ludvigson (2001), Zhang (2003) and Petkova and Zhang (2003).

The million dollar question arises as to discard CAPM or not! A meticulous analysis offers a solution. If we go back in time to the very basics as to how an investor defines and assigns value to risk perhaps we can find a solution. CAPM basics imply that an investor care for risk in Markowitz’s (1952) Mean-Variance (MV) framework. However there are two major problems associated to it. Firstly investors care more for downside risk as compared to upside risk. And secondly, to validate CAPM “normality” is assumed which is very much questionable. The irony is that you propose something and belief something works better. This is what happened to Markowitz himself. Initially, he is convinced that downside risk as semivariance is more appropriate as compared to variance as a measure of investor’s risk. However he stays with the variance but does recognize the importance of semivariance (Markowitz (1959)).

3.1 Are Downside Risk and Normality Really Fundamental

Challenging the theoretical MV-framework of CAPM is altogether not simple. The first major premise is that investors are safety cognizant and the second one is the empirical tests are based on condition on normal distribution. However the latter problem can be addressed by using Generalized Method of Moments (GMM) as normal distribution is a sufficient assumption which is assumed for statistical purposes and to make it empirically testable (Campbell, Lo and MacKinlay (1997), chapter 5). But still this condition does affect the empirical results and

---


29 He discards it due to lack of resources at that time. Variance is computationally simpler as semivariance optimization model requires twice the number of data inputs than the variance model and with the lack of cost-effective computer power until advent of the microcomputer in 1980s and with the fact that variance model is already mathematically very complex.

30 For detail , please see Abbas (2011)
though one can make CAPM testable but one cannot forget message it is conveying as to the investor’s preference for the downside utility function.

Downside risk is initially discarded by Markowitz (1952, 1959) and lived for three decades under the shadow of CAPM but it slowly makes its way to the arena. Roy (1952) starts the debate of downside risk and argues that investor care for disaster and safety from that disaster is the foremost goal of an investor thus entailing safety-first rule (SF-rule). Taken true means that Markowitz MV-framework and Roy’s SF-rule do not have any common ground to hold except under the condition of “normality” which is hard to find (Fama (1965a&b), Kon (1984), Affleck-Graves and McDonald (1989), Richardson and Smith (1993), Susmel (2001), Hwang and Pedersen (2002), Dufour, Khalaf and Beaulieu (2003)). In addition, Aparicio and Estrada (1997) study Scandinavian markets and assert that fat tails and high peaks is shown by distributions of daily stock returns. Bekaert and Harvey (1995, 1997), Eftekhari and Satchell (1996) and Bekaert, Erb, Harvey, and Viskanta (1998), show that emerging market display non-normality.

Researchers follow a pragmatic approach and prefer normal distribution being not a necessary condition to derive CAPM primarily assumed for statistical purposes. However under large departure from normality particularly when the distribution is severely asymmetric, MV-criterion may fail to correctly approximate the expected utility (see Chunachinda et al. (1997), Athayde and Flóres (2004) and Jondeau and Rockinger (2006)). A large body of literature is available purposing that MV-framework lacks theoretical support and stresses higher moments (skewness and kurtosis) are to be included.


Investors are indifferent to upside and downside risk is not true. They do not give equal weights to both upside and downside risk as assumed in CAPM. Markowitz (1959) himself agrees to it and Kahneman and Tversky (1979), Gul (1991) and Estrada (2000, 2002 and 2007) give reinforcement. Post and Levy (2005) emphasize that if investors show different behavior for bear and bull markets, then they are willing to pay premium for stocks giving downside protection in bear markets and upside potential in bull markets. Under this assumption one wonders how still CAPM survives the blitz directed to it all these decades with variance as the measure of risk.

---

31 Both Nielsen (1990) and Allingham (1991) provide sufficient conditions but these conditions are hard to interpret. Also see Berk (1997).
32 CAPM rests on MV-framework.
33 Downside risk is different from co-skewness risk. The downside risk is captured non-linearly while co-skewness does not assume it.
34 Especially for skewness and kurtosis.
35 Investors are risk averse for losses and they become seekers for gains.
Quirk and Saposnik (1962), Mao (1970), Klemkosky (1973) and Ang and Chua (1979) all demonstrate that semivariance is superior to variance. Grootveld and Hallerbach (1999) conclude that downside risk approaches tend to favor more as compared to variance. Harvey (2000) suggests the supremacy of downside risk for emerging markets for a sample based on equilibrium as well as non-equilibrium-based risk measures. Estrada (2002) uses methodology of Levy and Markowitz (1979), asserts credibility of semivariance over standard deviation-empirically. Balzer (2001) comes up with the relationship between utility and downside risk measures, and contests that semivariance framework is better than Markowitz’s M-V framework. Ang, Chen and Xing (2002, 2006) conclude that downside beta works better in predicting future than variance-based framework. Estrada (2002) affirms that over 45% of variability for a cross-sectional analysis for a joint sample of developed markets and emerging markets and 55% in emerging market are explained by DCAPM. Post and Vliet (2004) contest that DCAPM surpasses CAPM. Estrada and Serra (2005) find out global downside beta is the best risk measures among the various risk measures studied. This evidence is just the beginning, ahead lays vast frontiers of downside risk in form of its conditional and intertemporal versions. The basic premise of this debate is to establish downside risk the heir of CAPM in field of asset pricing but the question arises as to how downside risk is measured in asset returns.

3.2: Lower Partial Moment-Proxy for Downside Risk

Researchers though advocate supremacy of downside risk over variance as a measure of investors’ risk but face an uphill task of an appropriate proxy for downside risk in asset returns. Markowitz (1952, 1959) and Roy (1952) stress semivariance but the question arises how to measure downside risk! Markowitz has two suggestions for measuring downside risk: SVm- semivariance from mean return or below-mean semivariance and SVt- semivariance from a target return or below-target semivariance. Markowitz names these measures partial or semi-variances. In addition, Hogan and Warren (1974) substitute variance with semivariance but it fails to get due attention until the adoption of stochastic dominance in downside risk.

Stochastic dominance (SD) is a powerful risk analysis tool which does not rely on any certain type of distribution but, rather compares results in pairs using their cumulative distribution functions. In portfolio science, an investor exhibit three type of behavior namely; investor prefers more to less (nonsatiation), investor is risk averse and prefer more wealth but at a decreasing rate (decreasing absolute risk aversion (DARA)). Quadratic utility is assumed for variance in MV-framework using Taylor series to expand the utility function including the first two moments; mean and variance and ignoring the rest. However quadratic concave utility curve turns down after they reach the “bliss point”-certainly against the third assumption of the type of investor i.e. DARA. Under these circumstances SD seems to be the ideal tool as it does not

36 Proxy for downside risk.
37 MV-based CAPM primarily presumes risk-averse behavior of an investor.
38 Ang, Chen, and Xing (2002), Post and Vliet (2004) and Galagedera and Jaapar (2009) test conditional versions for both CAPM and downside risk based CAPM and conclude the superiority of the latter. Bali, Demirtas and Levy (2009) examine the comparison intertemporally, using VaR as a proxy for downside risk and their results indicate the dominance of VaR over variance and conditional variance.
39 “Bliss point” is a quantity of consumption such that further increases would make the consumer less satisfied (B. Binger and E. Hoffman (1997)).
40 DARA is one of many classes of Arrow-Pratt measure of absolute risk-aversion (ARA). Among them are IARA (increasing absolute risk aversion), CARA (constant absolute risk aversion), CRRRA (constant relative risk aversion) and DRRA/IRRA (decreasing/increasing relative risk aversion) are used (Arrow (1965) and Pratt (1964)).
assumes a specific form of distribution and the different form of SD capture all the three behaviors of an investor. SD uses probability distribution functions (pdf) of assets in pairs and asserts the dominance of one over the other. For the first condition SD forms First-order SD (FSD) such as first difference of a utility function is greater than zero i.e. $U'(w) > 0$. For the first two conditions SD assumes Second-order SD (SSD) as second difference be less than zero i.e. $U''(w) < 0$ and to fulfill all the three conditions SD assumes Third-order SD (TSD) as the third difference is positive i.e. $U'''(w) > 0$. The procedure seems simple and can be used both in portfolio optimization as well as for downside risk in asset pricing but the problem to quantify SD remains until the breakthrough by Bawa.

In 1975, Bawa develops proxy for SD. He generalizes semi-variance measure of risk with the development of the Lower Partial Moment (LPM), proxy for stochastic dominance, to reflect less restrictive class of decreasing absolute risk aversion (DARA) utility function. Bawa uses n-order LPMs to cover a wide range of risk measures as:

$$LPM_n(\tau, R_i) = \int_{-\infty}^{\tau} (\tau - R_i)^n dF(R_i)$$

where $\tau$ is the target return specified by the investor, $R_i$ is the return of asset $i$, $dF(R_i)$ is probability density function of asset $i$ and $n$ is the order of investor’s risk preference for ex ante. For setting $LPM=0$ (i.e. $n=0$) means that the investor is risk neutral. For $n=1$ means that the investor is risk averse but in a mean-variance framework. For $n=2$ implies that the investor is risk averse but from the point of semivariance. For $n=3$ and $n=4$, represent skewness and kurtosis. For discrete form the equation (1) takes the following form:

$$LPM_n(\tau, R_i) = \frac{1}{\tau} \sum_{i=1}^{\tau} \left[ \text{Max}(0, (\tau, R_i)) \right]^n$$

Fishburn (1977) extends the general LPM model into unlimited view of LPM which embraces all classes of investors; risk averse, risk seeking and risk neutral. For value $n<1$ captures risk seeking behavior while setting equal to zero is risk neutral behavior and risk averse behavior is $n > 1$. The higher the n (but above one), the higher the degree of risk aversion for an investor. Moreover with fractional degrees like 2.33 or 3.89 for different $n$ in Fishburn classes, LPM simply encompasses unlimited view of risk measures (Nawrocki (1999)).

Bawa and Lindenberg (1977) generalize LPM based on co-semivariance measure into generalized or asymmetric co-LPM or GCLPM for n-degree LPM structures defined as:

$$GCLPM_{ij,n}(\tau, R_i, R_j) = \int_{-\infty}^{\tau} \int_{-\infty}^{\tau} (\tau - R_i)^{n-1}(\tau - R_j)dF(R_i, R_j)$$

where normally the covariance between security $i$ is not equal to $j$ as depicted in Markowitz’s EV-framework and $dF(R_i, R_j)$ is the joint probability density function of returns of asset $i$ and $j$. However if one wishes to equal them then it has the flexibility to do so. In this case it will not be asymmetric co-LPM but symmetric co-LPM as follows:

$$GCLPM_{ij,n}(\tau, R_i, R_j) = GCLPM_{n}(\tau, R_j, R_i)$$

---

41 Quadratic utility also assumes IARA and IRRA while evidence is in favor of CRRA (Blume and Friend (1973)) and DARA (Cohn et al. (1975)). For detail, see Chavas (2004), chapter 4.
42 For details see Levy (2006).
43 For details see Elton, Gruber, Brown and Goetzmann (2003).
44 For $n \geq 0$ is FSD, $n \geq 1$ is SSD and $n \geq 2$ is TSD-efficient set. For details see Fishburn (1977) and Nawrocki (1991, 1999).
for asymmetric co-LPM,
\[ GCLPM_{ij,n} (\tau, R_i, R_j) = GCLPM_n (\tau, R_j, R_j) \].................................(5)

for symmetric co-LPM, in this the equation(5) will reduce to:
\[ GCLPM_{ij,n} (\tau, R_i, R_j) = LPM_n (\tau, R_i) \].................................(6)

when \( R_i = R_j \).

For ex post or discrete form, equation (4) takes the following form:
\[ GCLPM_{ij,n} (\tau, R_i, R_j) = \frac{1}{n} \sum_{t=1}^{n} [\text{Max}(0, (\tau, R_i))] \tau^{-1} (\tau - R_{ij})] \].............(7)

In all of above cases, Bawa and Lindenberg (1977) propose that \( \tau \) is equal to risk-free interest rate. However later versions became more flexible and \( \tau \) can assume any value according to investor’s choice\(^{45}\).

3.3 Algorithms for Lower Partial Moment

In a Markowitz world, quadratic programming is used to optimize a portfolio with linear constraints\(^ {46}\). As the risk of portfolio is defined as:
\[ \sigma_p^2 = \sum_{i=1}^{n} W_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j \sigma_{ij} \].................................(8)

where \( \sigma_p^2 \) is the variance of portfolio, \( W_i \) and \( W_j \) are weights of individual securities \( i \) and \( j \) and \( \sigma_{ij} \) is the covariance between securities \( i \) and \( j \). However it is assumed that under efficient diversification all risk is diversified but the covariance risk cannot be diversified (Sharpe (1964) and Elton, Gruber, Brown and Goetzmann (2003)) so all attention is paid to minimize the following expression:
\[ \text{Minimize } F(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} \].................................(9)

Subject to the following constraints:
\[ C1: R_p = \sum_{i=1}^{n} x_i (\bar{R}_i - R_p) \]
\[ C2: \sum_{i=1}^{n} x_i = 1 \]
\[ C3: x_i \geq 0, \ i = 1, 2, ..., N. \]

where \( x_i \) is portion of portfolio in security \( i \), \( R_p \) is portfolio return and \( R_i \) is return of security . \( C1 \) entails that the portfolio return is equivalent to weighted returns of individual securities. \( C2 \) implies that all weights should equal to 1 and \( C3 \) says that all weights should be more that 0 for no short sales allowed. However if short sales are allowed then \( C3 \) does not stand.

For downside risk maximization, two approaches can be used; Harlow (1991) applies (9) on LPM using the same constraints as discussed above:

\(^{46}\) For details see Elton, Gruber, Brown and Goetzmann (2003).
Minimize \( LPM_m(\tau;x) = \sum_{R_{m\tau}} p_p(\tau - R_p)^n \) \( \text{.........................(10)} \)

where \( p_p \) is the probability return of portfolio \( P \).

Harlow (1991), however, uses \( n = 1 \) or \( 2 \) and assumes co-LPM of security \( i \) equal to co-LPM of security \( j \)\(^{47}\). As the data depicts non-normality and asymmetry, so covariance of \( i \) and \( j \) cannot be equal to \( j \) and \( i \). Nawrocki (1991) propose another solution to downside risk minimization using LPM. He uses semi-deviation (SmD) instead of standard deviation to calculate co-LPM as follows:

\[
SmD_{ni} = \left\{ \frac{1}{t} \sum_{i=1}^{t} [\text{Max}(0, \tau - R_n)]^{1/n} \right\} \text{.........................(11)}
\]

and co-LPM as:

\[
GCLPM_{ij} = (SmD_{ni})(SmD_{nj})(\rho_{ij}) \text{.................................(12)}
\]

where \( \rho_{ij} \) is the correlation between asset \( i \) and \( j \). But the problem with it is that it assumes symmetry while the evidence is counter-symmetry. Under these circumstances, to modify (9) seems to be a feasible solution into the following function:

\[
\text{Minimize } F(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j GCLPM_{ij}(\tau, R_j, R_j) \text{.........................(13)}
\]

where

\[
GCLPM_{ij,n}(\tau, R_i, R_j) = \frac{1}{t} \sum_{i=1}^{t} [\text{Max}(0, (\tau, R_n))]^{n-1}(\tau - R_p)
\]

### 3.4 Matrices for Lower Partial Moment

The use of matrices has become very common in finance. As more and more computation is being done, matrices have become as essential ingredient for a financial econometrics researcher\(^{48}\). To solve the following function, matrices come very handy for a researcher:

\[
\text{Minimize } F(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j GCLPM_{ij}(\tau, R_j, R_j)
\]

Let \( W \) be \( n \times 1 \) vector matrix defined as:

\[
\text{Transpose } W = \{ w_i, w_j, \ldots, w_n \}
\]

where \( w_i, w_j \) and \( w_n \) are weights for securities \( i, j \) to \( n \). The co-LPM or GCLPM matrix is defined as \( n \times n \) matrix defines as:

\[
V = \begin{pmatrix}
GCLPM_{11} & \cdots & GCLPM_{1n} \\
\vdots & \ddots & \vdots \\
GCLPM_{n1} & \cdots & GCLPM_{nn}
\end{pmatrix}
\]

To calculate variance of a portfolio, we use the following expression:

\[
\text{Minimize } F(x) \text{ or } \sigma^2_p = [\text{Transpose}W] [V] [W] \text{.........................(14)}
\]

\(^{47}\) This makes Harlow’s LPM quite similar to Markowitz’s EV-framework as both assume covariance and security \( i \) and \( j \) equal to covariance of \( j \) and \( i \).

\(^{48}\) Portfolio analysis, yield curves, arbitrage with risk-free bonds, binomial option pricing, put-call parity and arbitrage pricing theory use matrices, for a detail discussion consult Teall (1999).
Minimize \( LPM_{m}(\tau; x) = \sum_{\tau_{j} < \tau} p_{p}(\tau - R_{p})^{n} \) \( \) ........................................(10)

where \( p_{p} \) is the probability return of portfolio \( P \).

Harlow (1991), however, uses \( n = 1 \) or \( 2 \) and assumes co-LPM of security \( i \) equal to co-LPM of security \( j \). As the data depicts non-normality and asymmetry, so covariance of \( i \) and \( j \) cannot be equal to \( j \) and \( i \). Nawrocki (1991) propose another solution to downside risk minimization using LPM. He uses semi-deviation \( (SmD) \) instead of standard deviation to calculate co-LPM as follows:

\[
SmD_{ni} = \left\{ \frac{1}{t} \sum_{t=1}^{t} [\text{Max}(0, \tau - R_{ni})]^{n} \right\}^{\frac{1}{n}}
\]  ........................................(11)

and co-LPM as:

\[
GCLPM_{ij} = (SmD_{ni}) (SmD_{nj}) \rho_{ji} \) .................................................(12)

where \( \rho_{ji} \) is the correlation between asset \( i \) and \( j \). But the problem with it is that it assumes symmetry while the evidence is counter-symmetry. Under these circumstances, to modify (9) seems to be a feasible solution into the following function:

\[
\text{Minimize } F(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \text{GCLPM}_{ij}(\tau; R_{i}, R_{j}) \]  ..............................................(13)

where

\[
\text{GCLPM}_{ij}(\tau; R_{i}, R_{j}) = \frac{1}{t} \sum_{j=1}^{t} [\text{Max}(0, (\tau, R_{ij}))]^{n} (\tau - R_{ji})
\]

3.4 Matrices for Lower Partial Moment

The use of matrices has become very common in finance. As more and more computation is being done, matrices have become as essential ingredient for a financial econometrics researcher. To solve the following function, matrices come very handy for a researcher:

\[
\text{Minimize } F(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \text{GCLPM}_{ij}(\tau; R_{i}, R_{j})
\]

Let \( W \) be \( n \times 1 \) vector matrix defined as:

\[
\text{Transpose } W = \{ w_{i}, w_{j}, \ldots, w_{n} \}
\]

where \( w_{i}, w_{j}, \) and \( w_{n} \) are weights for securities \( i, j \) to \( n \). The co-LPM or GCLPM matrix is defined as \( n \times n \) matrix defines as:

\[
V = \begin{pmatrix}
GCLPM_{11} & \ldots & GCLPM_{1n} \\
\vdots & \ddots & \vdots \\
GCLPM_{n1} & \ldots & GCLPM_{nn}
\end{pmatrix}
\]

To calculate variance of a portfolio, we use the following expression:

\[
\text{Minimize } F(x) \text{ or } \sigma_{p}^{2} = [\text{Transpose}W] [V] [W] \]  ........................................(14)

\[47\] This makes Harlow’s LPM quite similar to Markowitz’s EV-framework as both assume covariance and security \( i \) and \( j \) equal to covariance of \( j \) and \( i \).

\[48\] Portfolio analysis, yield curves, arbitrage with risk-free bonds, binomial option pricing, put-call parity and arbitrage pricing theory use matrices, for a detail discussion consult Teall (1999).
Excel-Solver or any other quadratic programming tool can be used to solve (14) and obtain different for each individual security to minimize the variance of a portfolio for a given portfolio return.

3.5 Lower Partial Moment Relation to Downside Beta

LPM-based risk measures are to be tested and to test it one comes back to square one i.e. CAPM. Tests of CAPM are to be applied if one has to compare results of regular-beta CAPM against downside-beta CAPM. LPM, a proxy of downside risk is not totally unrelated to other proxies of downside risk, namely the downside-beta. Harlow and Rao (1989) establish the relationship but first an understanding of regular-beta as follows is imperative:

\[ \beta = \frac{\text{cov}(R_i, R_m)}{\text{var}(R_m)} \]

where \( \text{cov}(R_i, R_m) \) is covariance of security \( i \) with market return and \( \text{var}(R_m) \) is variance of market return while downside-beta is defines as:

\[ \beta_d = \frac{\text{cov}(R_i, R_m | R_m < \tau)}{\text{var}(R_m | R_m < \tau)} \]

Harlow and Rao (1989), using \( \tau \) equal to risk-free rate define downside beta in terms of GCLPM and LPM as follows:

\[ \beta_d = \frac{GCLPM_d(\tau, R_i : R_m, R_f)}{LPM_d(\tau, R_i : R_m)} \]

However, \( \tau \) can be varied as discussed above and can be assigned any value while \( n \) is valued from 1 to 5 and the highest value used by Fishburn is 4.6 for an individual utility function (Fishburn (1977) and Nawrocki (1991)). Using (17) makes it easier to test downside-risk by applying conventional statistical test of CAPM and compare the two models\(^{49}\). Now the beta can be replaced by downside beta to test and compare the results of the two betas as follows\(^{50}\):

\[ E(R_i) = R_f + \beta_{d_i}[E(R_m) - R_f] \]

4. Conclusion

CAPM has always allured its audience right from its inception. However it is not free from controversies and the primary are that the investor places different weights to upside and downside risk-being more safety conscious and the model revolves around normality but is usually non-normal. On the other hand, CAPM popularity and high usage cannot be denied at all. A complete renovation is necessary only when needed and CAPM does not need it.

Instead of using variance as a measure of risk, downside risk is used. Evidence around the world is collected and is found to be superior to the conventional measure of risk. Next the theoretical framework is explored for downside risk with lower partial moment being a proxy to it. Lower partial moment, a proxy of stochastic dominance, is discussed with its both symmetric and asymmetric versions. Its strong points are that it accommodates investor’s preference of being safety-conscious and flexible to adopt non-normality. Lower partial moment algorithms are viewed and eventually its relation to downside beta is established.

\(^{49}\) Like Black, Jensen and Scholes (1972) and Fama and MacBeth (1973).

\(^{50}\) For details see Harlow and Rao (1989).
Research does not end here but it leads to other worlds of consumption and intertemporal models and discrete and continuous models. With high inception of statistical techniques like GMM, GARCH and Markov Switching Models into field of finance, results can be made better, and enhanced versions of downside risk and lower partial moment can be developed which can be tested with accuracy and commercialized as risk measurement techniques in the field of asset pricing.
References


Aparicio, Felipe Miguel and Estrada, Javier (April 1997), Empirical Distributions of Stock Returns: European Securities Markets, 1990-95


Shanken (1987), 'Nonsynchronous Data and the Covariance-Factor Structure of Returns', Journal of Finance, 42 (2), June