On some classes of $p$-valent functions involving Carlson–Shaffer operator

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ABSTRACT

Let $A(p)$, $p \in \mathbb{N}$, be a class of functions $f : f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ analytic in the open unit disc $E$. We use Carlson–Shaffer operator for $p$-valent functions to define and study certain classes of analytic functions. Inclusion results, a radius problem and some other interesting properties are discussed.

1. Introduction

Let $A(p)$ denote a class of functions $f$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \ldots\}), \quad (1.1)$$

which are analytic and $p$-valent in the open unit disc $E = \{z : |z| < 1\}$.

A function $f \in A(p)$ is said to belongs to the class $S'(p, \rho)$, $0 \leq \rho < p$, if and only if, $\Re \frac{zf'(z)}{f(z)} > \rho$, $z \in E$ and with $z = re^{i\theta}$,

$$\int_{0}^{2\pi} \Re \left( \frac{zf'(z)}{f(z)} \right) d\theta = 2\rho \rho.$$ 

$f \in S'(p, \rho)$ is called $p$-valent starlike function of order $\rho$. We can define a class $C(p, \rho)$ of $p$-valent convex functions. A function $f \in A(p)$ belongs to $C(p, \rho)$ if and only if $\frac{zf'(z)}{f(z)} \in S'(p, \rho)$, for $z \in E$. See also [1–3].

For two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p, n \in \mathbb{N}),$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Let the function $\phi_p(a, c; z)$ be defined by

$$\phi_p(a, c; z) = z^p + \sum_{k=p+1}^{\infty} \frac{(a)}{c_k} z^k, \quad (1.2)$$

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where \( z \in E, a \) is real, \( c \neq 0 \), \( 1, -1, -2, -3, \ldots \). In (1.2), we use Pochhammer symbol \( (\lambda)_k \) or the shifted factorial which is defined as

\[
(\lambda)_0 = 1, \quad (1)_k = k! \quad (k \in N_0 = N \cup \{0\}),
\]

and

\[
(\lambda)_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) \quad (k \in N).
\]

Corresponding to the function \( \phi_p(a, c; z) \) given by (1.2) a linear operator \( L_p(a, c) \) is defined [4,5] by using Hadamard product (or convolution) as follows

\[
L_p(a, c)f(z) = \phi_p(a, c; z) * f(z) \quad (f \in A(p)),
\]

or equivalently

\[
L_p(a, c)f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(a)_k}{(c)_k} a_k z^k \quad (z \in E).
\]

The definition (1.3) or (1.4) of the linear operator \( L_p(a, c) \) is motivated essentially by the well-known Carlson–Shaffer operator \( L(a, c) = L_1(a, c) \), see [6].

We note that, for \( f \in A(p) \),

\[
L_p(n + p, 1)f(z) = \frac{z^p}{(1 - z)^{n+p}} * f(z) = D^{n+p-1}f(z),
\]

where \( n > -p \). The operator \( D^{n+p-1} \), for \( p = 1 \), was introduced by Ruscheweyh [7] and the symbol \( D^{n+p-1} \) was introduced in [8], as the Ruscheweyh derivative of \( (n + p - 1) \)th order.

From (1.3), we have the easily derived familiar identity:

\[
z(L_p(a, c)f(z))' = aL_p(a + 1, c)f(z) - (a - p)L_p(a, c)f(z).
\]

We note that

\[
L_p(a, a)f(z) = f(z), \quad L_p(p + 1, p)f(z) = \frac{zf'(z)}{p}.
\]

Let \( P_0(\rho) \) be the class of functions \( h(z) \), analytic in the unit disc \( E \), satisfying the properties \( h(0) = 1 \) and

\[
\int_0^{2\pi} \left| \text{Re} h(z) - \rho \right| d\theta \leq k\pi,
\]

where \( z = re^{i\theta} \) and \( k \geq 2 \). \( 0 < \rho < p \). This class was introduced in [9]. We note that \( P_0(0) = P_0 \) which has been studied in [10]. Also \( P_2(\rho) = P(\rho) \), the class of analytic functions with positive real part greater than \( \rho \) and \( P_2(0) = P \), the class of functions with positive real part.

From (1.7) and using the technique of Pinchuk [10] for the class \( P_0(0) \), one can easily have

\[
h(z) = \left( k - \frac{1}{2} \right) h_1(z) - \left( k - \frac{1}{2} \right) h_2(z), \quad h_1, h_2 \in P(\rho) \quad \text{and} \quad z \in E.
\]

We now define the following.

**Definition 1.1.** Let \( f \in A(p) \). Then \( f \in S'(a, c, p, \rho) \) if and only if

\[
\left\{ \begin{array}{l}
z[L_p(a, c)f(z)]' \\
L_p(a, c)f(z)
\end{array} \right\} \in P(\rho) \quad (z \in E).
\]

It is obvious that \( f \in S'(a, c, p, \rho) \) if and only if \( L_p(a, c)f \in S'(p, \rho) \) for \( z \in E \).

We also note that:

(i) \( S'(n + 1, 1, p, \rho) = T_{n+p, 1}(\rho) \), a class of \( p \)-valent functions studied in [11],

(ii) \( S'(n + 1, 1, 1, \rho) = S'(n, \rho) \), see Noor [12].

**Definition 1.2.** Let \( 0 < \rho < p \), \( \mu > 0 \); \( a > 0 \), \( c \) be any real numbers other than \( 0, -1, -2, -3, \ldots \) and let \( \lambda \) be a complex number such that \( \text{Re} \lambda > 0 \). Then a function \( f \in A(p) \) is said to be in the class \( B_0^a(a, c, p, \mu, \delta) \) if and only if it satisfies

\[
\left\{ (1 - \lambda) \frac{L_p(a, c)f(z)^\mu}{L_p(a, c)g(z)} + \lambda \frac{L_p(a + 1, c)f(z)}{L_p(a + 1, c)g(z)} \right\} \in P_1(\rho),
\]

where \( k \geq 2 \), \( z \in E \) and \( g \in A(p) \) satisfies the condition

\[
\beta(z) = \left[ \frac{L_p(a + 1, c)g(z)}{L_p(a, c)g(z)} \right] \in P(\delta) \quad (0 \leq \delta < p, \ z \in E).
\]

\[
\beta(z) = \left[ \frac{L_p(a + 1, c)g(z)}{L_p(a, c)g(z)} \right] \in P(\delta) \quad (0 \leq \delta < p, \ z \in E).
\]
From the above definition, the following subclass of $A(p)$ and $A(1)$ emerge as special cases.

(i) When $a = n + p(n > -p)$, $c = 1$, $k = 2$, we have $B_{p}^{0}(n + p, 1, p, \rho, \mu, \delta)$, a subclass of $A(p)$ studied in [13].

(ii) For $a = c$, $k = 2$, $\lambda = 0$, $\mu = p = 1$, $g(z) = z$, we have

$$B_{p}^{0}(a, a, 1, \rho, 1, 1, 1) = \left\{ f \in A(1) : \frac{f(z)}{z} \in P(\rho), \ 0 \leq \rho < 1, \ z \in E \right\},$$

a class studied by Chen [14].

(iii) When $a = c = p = 1$, $g(z) = z$, the class $B_{p}^{0}(a, c, p, \rho, \mu, \delta)$ reduces to a class studied by Noor [15].

(iv) For $a = c = p = \lambda = 1$, $k = 2$ and $g(z) = z$, the class $B_{p}^{0}(a, c, p, \rho, \mu, \delta)$ reduces to the class

$$B_{p}^{1}(1, 1, 1, 1, 0, 1) = B_{1}(\rho, 1) = \left\{ f \in A(1) : \frac{zf'(z)}{f(z)} \in P(\rho), \ 0 \leq \rho < 1, \ z \in E \right\},$$

studied in [16].

(v) If we take $a = c = \lambda = p = 1$, $k = 2$, $\rho = 0$, $g(z) = z$, then $B_{p}^{0}(a, c, p, \rho, \mu, \delta)$ reduces to $B_{1}(\mu)$, a subclass of class of Bazilevic functions investigated by Singh [17].

(vi) Let $a = n + 1$, $c = p = \lambda = 1$, $k = 2$ and $g(z) = z$. Then, we have the class

$$B_{p}^{0}(n + 1, 1, 1, \rho, 0, 0) = \left\{ f \in A(1) : \left( \frac{D^{n+1}f(z)}{z} \right)^{\mu-1} \in P(\rho), \ 0 \leq \rho < 1, \ z \in E \right\},$$

and this is a subclass of class of Bazilevic functions of type $\rho$.

2. Preliminary results

We list some preliminary lemmas required for proving our main results.

**Lemma 2.1** [18]. Let $u = u_{1} + iu_{2}$ and $v = v_{1} + iv_{2}$ and let $\Psi(u, v)$ be a complex-valued function satisfying the conditions:

(i) $\Psi(u, v)$ is continuous in a domain $D \subset \mathbb{C}^{2}$,

(ii) $(1, 0) \in D$ and $\Psi(1, 0) > 0$,

(iii) $Re \Psi(u_{2}, v_{1}) \leq 0$, whenever $(u_{2}, v_{1}) \in D$ and $v_{1} \leq -\frac{1}{2}(1 + u_{2}^{2})$. If $h(z) = 1 + \sum_{n=1}^{\infty}c_{n}z^{n}$ is a function analytic in $E$ such that $(h(z), zh'(z)) \in D$ and $Re \Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $Re h(z) > 0$ in $E$.

Using relation (1.8) and a result proved in [19], we can easily derive the following:

**Lemma 2.2.** If $h(z)$ is analytic in $E$ with $h(0) = 1$ and if $\lambda_{1}$ is a complex number satisfying $Re \lambda_{1} > 0$, then $\{h(z), \lambda_{1}zh'(z)\} \in P_{k}(\rho)(0 \leq \rho < p)$ implies $h(z) \in P_{k}(\gamma)$, where $\gamma$ is given by

$$\gamma = \rho + (p - \rho)(2\gamma_{1} - 1), \gamma_{1} = \int_{0}^{1} (1 + te^{\lambda_{1}t})^{-1} dt, \quad (2.1)$$

with $\gamma_{1} \in [1, p]$, an increasing function of $Re(\lambda_{1})$ and $\frac{1}{2} \leq \gamma_{1} < p$. The estimate is sharp in the sense that the bound cannot be improved.

**Lemma 2.3** [20]. If $h(z)$ is analytic in $E$ with $h(0) = 1$ and $h \in P_{k}(\gamma)$ for $z \in E$, then, for any function $F$ analytic in $E$, the function $h \ast F$ takes values in the convex hull of the image of $E$ under $F$.

3. Main results

**Theorem 3.1.** Let for $\lambda \geq 0$ and $f \in B_{p}^{0}(a, c, p, \rho, \mu, \delta)$. Then $f \in B_{p}^{0}(a, c, p, \rho_{1}, \mu, \delta)$, where $\rho_{1}$ is given by

$$\rho_{1} = \frac{2\mu \rho \lambda + \lambda \rho \delta}{2\mu \rho + \lambda \delta} \quad (3.1)$$

**Proof.** Let

$$h(z) = (p - \rho_{1})^{-1} \left\{ \left( \frac{L_{p}(a, c, f(z))}{L_{p}(a, c, g(z))} \right)^{\mu} - \rho_{1} \right\} = \left( \frac{k}{4} + \frac{1}{2} \right)h_{1}(z) - \left( \frac{k}{4} - \frac{1}{2} \right)h_{2}(z). \quad (3.2)$$

where $h$ is analytic in $E$ with $h(0) = 1$ and $g$ satisfies condition (1.10).
From (3.2), together with (1.6), (1.8), (1.10) and some computations, we have
\[
(1 - z_j) \left( \frac{L_p(a, c) f(z)}{L_p(a, c) g(z)} \right)^{\mu} + z_j \left( \frac{L_p(a + 1, c) f(z)}{L_p(a + 1, c) g(z)} \right)^{\mu-1} = (p - \rho) \frac{h(z)}{\mu a(z)} + \frac{z_j(p - \rho_1)z h'(z)}{\mu a(z)}
\]
\[
= \left( \frac{k}{4} + \frac{1}{2} \right) \left( p - \rho \right) h_i(z) + \left( p - \rho_1 \right) h_i(z) + \frac{z_j(p - \rho_1)z h'(z)}{\mu a(z)}.
\]

Since \( f \in B_k^+(a, c, p, u, \delta) \), it follows that
\[
\left\{ (p - \rho) \frac{h(z)}{\mu a(z)} + (p - \rho_1) \frac{h(z)}{\mu a(z)} \right\} \in P(\rho), \quad z \in E, \quad i = 1, 2, \quad 0 \leq \rho < p.
\]
That is,
\[
\frac{1}{p - \rho} \left\{ (p - \rho) \frac{h(z)}{\mu a(z)} + (p - \rho_1) \frac{h(z)}{\mu a(z)} \right\} \in P, \quad i = 1, 2
\]
To prove our result, we want to show that \( h_i \in P, \ i = 1, 2 \) and \( z \in E \). We form the functional \( \Psi(u, v) \) by taking \( u = h_i(z) \), \( v = z h'(z) \). Thus we have
\[
\Psi(u, v) = (p - \rho_1)u + (p - \rho_1) + \frac{z_j(p - \rho_1)z h'(z)}{\mu a(z)}.
\]
(3.3)
The first two conditions of Lemma 2.1 are clearly satisfied. We proceed to verify the condition (iii) as follows:
\[
\Re \{\psi(it_2, v_1)\} = (p - \rho_1) - \Re \\left\{ \frac{z_j(p - \rho_1)u_1}{\mu a(z)} \right\} \leq (p - \rho_1) - \frac{\lambda(p - \rho_1)(1 + u_2)\delta}{2\mu a(z)^2} = 2\mu a(z)^2 \rho - \frac{\lambda(p - \rho_1)(1 + u_2)\delta}{2\mu a(z)^2} = A + B u_2^2.
\]
where
\[
A = 2\mu a(z)^2 \rho - \frac{\lambda(p - \rho_1)(1 + u_2)\delta}{2\mu a(z)^2}.
\]
B = \(-\lambda(p - \rho_1)\delta \leq 0.
\]
if \( 0 \leq \rho < p, \)
\[
C = \mu a(z)^2 > 0.
\]
From the relation (3.1), we obtain \( A \leq 0 \), and this gives us \( \Re \{\psi(it_2, v_1)\} \leq 0 \). Now, we use Lemma 2.1 to have \( h_i \in P, i = 1, 2 \) for \( z \in E \) and this completes the proof. □

Theorem 3.2. Let \( 0 \leq \lambda_1 < \lambda_2 \). Then \( B_k^+(a, c, p, \mu, \delta) \subset B_k^+(a, c, p, \mu, \delta) \).

Proof. Let \( f \in B_k^+(a, c, p, \mu, \delta) \). Then we have
\[
(1 - \lambda_2) \left( \frac{L_p(a, c) f(z)}{L_p(a, c) g(z)} \right)^{\mu} + \lambda_2 \left( \frac{L_p(a + 1, c) f(z)}{L_p(a + 1, c) g(z)} \right)^{\mu-1} = H_2(z) \in P_k(\rho) \quad (0 \leq \rho < p, \ z \in E).
\]
Now, by Theorem 3.1, we see that
\[
\left( \frac{L_p(a, c) f(z)}{L_p(a, c) g(z)} \right)^{\mu} = H_1(z) \in P_k(\rho_1) \subset P_k(\rho).
\]
Thus, for \( \lambda_3 \geq 0 \),
\[
\left\{ (1 - \lambda_3) \left( \frac{L_p(a, c) f(z)}{L_p(a, c) g(z)} \right)^{\mu} + \lambda_3 \left( \frac{L_p(a + 1, c) f(z)}{L_p(a + 1, c) g(z)} \right)^{\mu-1} \right\} = \left( 1 - \frac{\lambda_3}{\lambda_2} \right) H_1(z) + \frac{\lambda_3}{\lambda_2} H_2(z).
\]
(3.4)
Since the class \( P_k(\rho) \) is a convex set, see [21], it follows that the right hand sides of (3.4) belongs to \( P_k(\rho) \) for \( z \in E \). This implies that \( f \in B_k^+(a, c, p, \mu, \delta) \) and we obtain the required result. □

We define the operator \( J_c : A(p) \to A(p) \) as follows.
\[
J_c(f) = c + \frac{p}{2z} \int_0^z t^{-1} f(t) dt \quad (c > -p).
\]
(3.5)
For \( p = 1, c \in N \), the operator \( J_c(f) \) was introduced by Bernardi [22] and, in particular, \( J_1(f) \) was studied by Libera [23] and Livingston [24].
Theorem 3.3. Let \( f \in A(p) \) and \( f_n(f) \) be given by (3.5). If
\[
\left\{ (1 - \lambda) \frac{L_p(a,c)f(z)}{z^p} + \lambda \frac{L_p(a,c)f(z)}{z^p} \right\} \in P_k(\rho),
\]
then
\[
\left\{ L_p(a,c)f(z) \right\} \in P_k(\gamma) \quad (z \in E),
\]
where \( \gamma \) is given by (2.1) with \( \gamma = \frac{1}{\mu} \).

Proof. From (3.5), we can write
\[
z(L_p(a,c)f(z))' = (c + p)(L_p(a,c)f(z) - c(L_p(a,c)f(z)))
\]
Let \( \left\{ \frac{L_p(a,c)f(z)}{z^p} \right\} = h(z) \). Then, using (3.6), we have
\[
(1 - \lambda) \frac{L_p(a,c)f(z)}{z^p} + \lambda \frac{L_p(a,c)f(z)}{z^p} = \left\{ h(z) + \frac{\lambda h'(z)}{c + p} \right\} \in P_k(\rho).
\]
Now, applying Lemma 2.2, we obtain the required result. \( \square \)

We now deal with the converse case of Theorem 3.1 as follows.

Theorem 3.4. Let \( f \in B^0_a(a,c,p,\rho,\mu,0) \). Then \( f \in B^0_\mu(a,c,p,\rho,\mu,0) \) for \( |z| < R \), where
\[
R = \frac{\mu a + \lambda}{\alpha a}.
\]

Proof. Let \( f \in B^0_\mu(a,c,p,\rho,\mu,0) \). Then it follows that
\[
\left( \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} \right)^\mu = (p - \rho)h(z) + \rho,
\]
where \( h \in P_k \) and \( g \in A(p) \) satisfies the condition
\[
\beta(z) = \frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)} \in P \text{ in } E.
\]
From (3.8), (3.9) and (1.6), we have
\[
\frac{1}{p - \rho} \left\{ (1 - \lambda) \left( \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} \right)^\mu + \lambda \left( \frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} \right)^\mu - \rho \right\} = h(z)
\]
\[
+ \frac{\lambda h'(z)}{\mu a h(z)}, \quad \beta(z) \in P, \quad h \in P_k, \quad z \in E = \left( \frac{k}{4}, \frac{1}{2} \right) \left\{ h_1(z) + \frac{\lambda h'_1(z)}{\mu a h(z)} \right\} - \left( \frac{k}{4}, \frac{1}{2} \right) \left\{ h_2(z) + \frac{\lambda h'_2(z)}{\mu a h(z)} \right\},
\]
where we have used (1.8), \( h_i \in P, \quad i = 1,2, \quad \beta \in P, \quad z \in E \).

Now, using well known [1] estimates for the class \( P \),
\[
|zh_1'(z)| < \frac{2Re h_1(z)}{(1 - r^2)}
\]
\[
Re h_1(z) \geq \frac{1 - r}{1 + r} \quad (|z| < 1, \quad i = 1,2, \quad z \in E),
\]
we have
\[
Re \left\{ h_1(z) + \frac{\lambda h'_1(z)}{\mu a h(z)} \right\} \geq Re \left\{ h_1(z) - \frac{\lambda h'_1(z)}{\mu a h(z)} \right\} \geq Re h_1(z) \left[ 1 - \frac{2r}{\mu a(1 - r)^2} \right] = Re h_1(z) \left[ \frac{\mu a(1 - r)^2 - 2r}{\mu a(1 - r)^2} \right].
\]
The right hand side of the above inequality is positive, if \( r < R \), where \( R \) is given by (3.7). Sharpness of this result follows by taking \( h_1(z) = \frac{1}{1 - z^2} \). Consequently it follows form (3.10) that \( f \in B^0_\mu(a,c,p,\rho,\mu,0) \) for \( |z| < R \), where \( R \) is given by (3.6). \( \square \)

Special cases

(i) Let \( \mu = a = c = p = \lambda = 1, k = 2 \), and \( f \in B^0_2(1,1,1,\rho,0) \), for \( z \in E \). Then \( f \in B^0_2(1,1,1,\rho,0) \) for \( |z| < 2 - \sqrt{3} \approx 0.2679 \).

(ii) When \( \mu = c = \lambda = p = 1, k = a = 2, f \in B^0_2(1,1,1,\rho,0), z \in E, \) then \( f \in B^0_2(1,1,1,\rho,0) \), for \( |z| < \frac{2}{1 + \sqrt{3}} \approx 0.38197 \).

(iii) For \( \mu = a = c = p = \lambda = 1, k = k, f \in B^0_2(1,1,1,\rho,0) \), for \( z \in E \) implies \( f \in B^0_2(1,1,1,\rho,0) \), for \( |z| < 2 - \sqrt{3} \approx 0.2679 \).
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