Radii problems for certain analytic functions

Bushra Malik *

Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan

Received 19 June 2010; accepted 1 July 2010
Available online 1 August 2010

KEYWORDS
Bounded boundary rotation; Bounded radius rotation; Convolution; Ruscheweyh derivative

Abstract
In this paper, we study some radii problems for certain classes of analytic functions. These results generalize some of the previously known radii problems such as the radius of convexity for starlikness and radius of quasi-convexity for close-to-convex functions. Also, it is shown that some of these radii are best possible.

1. Introduction
Let $A_n$ be the class of analytic functions $f$,
\[ f(z) = z + \sum_{m=1}^{\infty} a_{m+1} z^{m+1}, \quad n \geq 1, \] (1.1)
which are analytic in the unit disc $E = \{ z : |z| < 1 \}$. We shall need the following known classes (Noor, 2008) in our discussion. Let, for $0 \leq \gamma, \beta < 1$,

\[ p(z) = 1 + \sum_{m=0}^{\infty} a_{m+1} z^{m+1}, \quad z \in E, \] (1.2)

we have,
\[ P(\gamma, n) = \{ p : p \text{ is analytic in } E, \text{ given by (1.2), } \Re p(z) > \gamma \} \]
\[ S'(\gamma, n) = \left\{ f : f \in A_n \text{ and } \frac{zf'}{f} \in P(\gamma, n) \right\} \]
\[ C(\gamma, n) = \left\{ f : f \in A_n \text{ and } \left( 1 + \frac{zf'}{f} \right) \in P(\gamma, n) \right\}. \]

It is clear that
\[ f \in C(\gamma, n) \iff zf' \in S'(\gamma, n). \] (1.3)

For $n = 1$, these classes have been introduced by Robertson (1963).
\[ K(\beta, \gamma) = \left\{ f : f \in A_n \text{ and } \frac{zf}{g} \in P(\beta, n) \text{ for some } g \in S'(\gamma, n) \right\} \]
\[ C'(\beta, \gamma) = \left\{ f : f \in A_n \text{ and } \frac{zf'}{g'} \in P(\beta, n) \text{ for some } g \in C(\gamma, n) \right\}. \]

We note here that
\[ f \in C'(\beta, \gamma) \iff zf' \in K(\beta, \gamma). \] (1.4)
These classes have been studied by Noor (1987) for \( n = 1 \).

Now, we have the definition of following classes for \( k \geq 2 \),

\[
P_k(\gamma, n) = \left\{ p : p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z), \quad p_1, p_2 \in P(\gamma, n) \right\}
\]

\[
P'_k(\gamma, n) = \left\{ p : p'(z) \in P_k(\gamma, n) \right\}
\]

\[
V_k(\gamma, n) = \left\{ f : f \in A_n \text{ and } \left( 1 + \frac{zf'}{f} \right) \in P_k(\gamma, n) \right\}
\]

\[
R_k(\gamma, n) = \left\{ f : f \in A_n \text{ and } \frac{zf'}{f} \in P_k(\gamma, n) \right\}
\]

We note that \( P_2(\gamma, n) = P(\gamma, n) \). The class \( P_1(0, 1) = P_k \) was introduced and discussed by Pinchuk (1971), where he defined it as follows:

\[
P_k = \left\{ p : p(z) = \frac{1}{2} \int_{-\infty}^{\infty} \left( 1 + \frac{ze^{-iu}}{1 - ze^{-iu}} \right) d\mu(t), \quad \times \int_{-\infty}^{\infty} d\mu(t) = 2, \quad \int_{-\infty}^{\infty} |d\mu(t)| \leq k \right\}
\]

It is easy to see that

\[
P_k(\gamma, n) = (1 - \gamma) P_k(0, n) + \gamma
\]

and \( P_2(0, 1) = P \) is the class of functions with positive real part. It is clear that

\[
f \in V_k(\gamma, n) \iff zf' \in R_k(\gamma, n).
\]

The classes \( V_k(0, 1) = V_k \) and \( R_k(0, 1) = R_k \) are the well-known classes of functions with bounded boundary rotation and bounded radius rotation, respectively.

Let \( f(z), j = 1, 2 \) in \( A_n \) be given by

\[
f_j(z) = z + \sum_{m=1}^{\infty} a_{m+1}z^{m+1}, \quad n \geq 1, \quad z \in E.
\]

Then the Hadamard product or convolution \((f_1 * f_2)(z)\) of \( f_1 \) and \( f_2 \) is defined by

\[
(f_1 * f_2)(z) = z + \sum_{m=0}^{\infty} a_{m+1}^{(1)} a_{m+1}^{(2)} z^{m+1}, \quad n \geq 1, \quad z \in E.
\]

By using the Hadamard product, we define the well-known Ruscheweyh derivative (see Ruscheweyh, 1975) as following.

Denote by \( D^x : A_n \rightarrow A_n \) the operator defined by

\[
D^x f(z) = \frac{z^m}{(1-z)^{m+1}} f(z), \quad m > -1.
\]

For \( x = m \in N_0 = \{ 0, 1, 2, \ldots \} \), we can write

\[
D^n f(z) = \frac{z}{(1-z)^{n+1}} f(z) = \frac{z(z-1)f(z)^{(n)}}{n!}.
\]

Also, for Ruscheweyh derivative \( D^x \), the following identity is known (see Fukui and Sakaguchi, 1980).

For a real number \( x > -1 \), we have

\[
z(D^x f(z))' = (z+1)D^{x+1} f(z) - zD^x f(z).
\]

Recently, the Ruscheweyh derivative has been studied in Noor and Hussain (2008).

We now have the following classes which have been introduced and studied in Noor (1991), for the case \( n = 1 \),

\[
S_\alpha(\gamma, n) = \{ f : f \in A_n : D^\alpha f(z) \in S(\gamma, n), \ \alpha > -1, \ z \in E \}
\]

\[
C_\alpha(\gamma, n) = \{ f : f \in A_n : D^\alpha f \in C(\gamma, n), \ \alpha > -1, \ z \in E \}
\]

\[
K_\alpha(\beta, \gamma) = \{ f : f \in A_n : D^\alpha f \in K(\beta, \gamma), \ \alpha > -1, \ z \in E \}
\]

and

\[
C_\alpha^*(\beta, \gamma) = \{ f : f \in A_n : D^\alpha f \in C^*(\beta, \gamma), \ \alpha > -1, \ z \in E \}.
\]

2. Preliminary results

The following lemmas will be used.

**Lemma 2.1.** Let \( h \in P(0, n) = P_n \) for \( z \in E \). Then

\[
\left| \frac{h(z)}{h(0)} \right| \leq \frac{2|z|^n + n|z|^{n-1}}{1 - |z|^n}
\]

\[
\left| z h'(z) \right| \leq \frac{2|z|^n + n|z|^{n-1}}{1 - |z|^n}
\]

\[
\frac{1 - |z|^n}{1 - |z|^n} \leq \text{Re} h(z) \leq |h(z)| \leq \frac{1 + |z|^n}{1 - |z|^n}.
\]

For (i) we refer to MacGraw (1963), (ii) will be found in Bernardi (1974) and for (iii) (see Shah, 1972).

The following Lemma can easily be shown by using **Lemma 2.1**.

**Lemma 2.2.** Let \( p \in P(\gamma, n) \) for \( z = re^{i\theta} \in E \). Then

\[
\frac{1 + (2\gamma - 1)r^n}{1 + r^n} \leq \text{Re} p(z) \leq |p(z)| \leq \frac{1 - (2\gamma - 1)r^n}{1 - r^n}
\]

and

\[
|zp'(z)| \leq \frac{2n(1 - \gamma)r^n |p(z)|}{(1 - r^n)[1 + (1 - 2\gamma)r^n]}.
\]

3. Main results

**Theorem 3.1.** Let \( f, g \in A_n \) and let \( \frac{f}{g} \in P_n \), where \( g \in V_k(\gamma, n) \). Then \( f \in V_k(\gamma, n) \) for \( |z| < r_0 \), where

\[
r_0 = r_0(\gamma, n) = \frac{1 - \gamma}{(n - \gamma + 1) + \sqrt{n^2 - 2\gamma n + 2n}}.
\]

This result is sharp.

**Proof.** We can write

\[
f(z) = g(z)h(z), \quad g \in V_k(\gamma, n), \quad h \in P_n, \quad z \in E.
\]

Logarithmic differentiation yields,

\[
\frac{z(\frac{g'(z)}{g(z)})'}{f'(z)} + \frac{zh'(z)}{h(z)} = \frac{H(z) + \frac{zh'(z)}{h(z)}}{h(z)}, \quad h \in P_k(\gamma, n)
\]

\[
= \left( \frac{k}{4} + \frac{1}{2} \right) ((1 - \gamma)H_1(z) + \gamma)
\]

\[
= \left( \frac{k}{4} + \frac{1}{2} \right) ((1 - \gamma)H_2(z) + \gamma) + \frac{zh'(z)}{h(z)},
\]

where \( H_i \in P_n, \ i = 1, 2 \).
This gives us
\[ \frac{1}{1 - \gamma} \left\{ \left( \frac{z f'(z)}{f(z)} \right)' - \gamma \right\} = \left( \frac{k + 1}{2} \right) \left\{ H_1(z) + \frac{1}{1 - \gamma} \frac{zh'(z)}{h(z)} \right\} \]
\[ - \left( \frac{k - 1}{2} \right) \left\{ H_2(z) + \frac{1}{1 - \gamma} \frac{zh'(z)}{h(z)} \right\}. \]

Now, for \( i = 1, 2 \), we use Lemma 2.1, with \( |z| = r \), to have
\[ \text{Re} \left\{ H_i(z) + \frac{1}{1 - \gamma} \frac{zh'(z)}{h(z)} \right\} \geq \text{Re} \left\{ H_i(z) - \frac{1}{1 - \gamma} \frac{zh'(z)}{h(z)} \right\} \]
\[ \geq 1 - r^\gamma - \frac{1}{1 + r^\gamma} \left( 1 - \frac{1}{1 - \gamma} (1 - r^{2\gamma}) \right) \]
\[ = \frac{(1 - \gamma)(1 - r^\gamma)^2 - 2\gamma r^\gamma}{(1 - \gamma)(1 - r^{2\gamma})} \]
\[ = \frac{(1 - \gamma) - 2(\gamma - 1) r^\gamma + (1 - \gamma) r^{2\gamma}}{(1 - \gamma)(1 - r^{2\gamma})}. \]

Therefore, for \( |z| < r_0 \) where \( r_0 \) is as stated in (3.1),
\[ \text{Re} \left\{ H_i(z) + \frac{1}{1 - \gamma} \frac{zh'(z)}{h(z)} \right\} > 0. \]

This implies that
\[ \frac{1}{1 - \gamma} \left\{ \left( \frac{z f'(z)}{f(z)} \right)' - \gamma \right\} \in P_\Phi(0, n) \quad \text{for} \quad |z| < r_0, \]

which leads us to the required result that \( f \in V_\Lambda(\gamma, n) \) for \( |z| < r_0 \), where \( r_0 \) is as given in (3.1).

Sharpness can be seen by taking
\[ H_i(z) = h(z) = \frac{1 - z^n}{1 + z^n} \in P_\Lambda. \]

**Theorem 3.2.** Let \( f, g \in A_\Lambda \) and \( g \in P_\Lambda(\gamma, n) = P(\gamma, n) \) in \( E \). If \( \frac{f}{g} \in P(\gamma, n) \), then \( f \in V_\Lambda(\gamma, n) \) for \( |z| < r_1 \) where \( r_1 \) is given by
\[ r_1^\gamma = \frac{1}{2n(1 - \gamma) + \sqrt{4n^2(1 - \gamma)^2 + 1}}. \]

This result is also sharp.

**Proof.** Since \( f \in S_\Lambda(\gamma, n) \), so we can write it as
\[ (z f(z))' = H(z) = h(z) + z \gamma, \]

where \( H \in P(\gamma, n) \) and so \( h \in P(0, n) = P_\Lambda \). Now using (1.7), we have
\[ \frac{D(f(z))'}{D(f(z))} = \frac{1}{1 + z} \left\{ \frac{D(f(z))'}{D(f(z))} + z \right\} = \frac{1}{1 + z} \left\{ (1 - \gamma) h(z) + z \right\}. \]

Differentiating both sides logarithmically, we have
\[ \frac{z(D(f(z))')'}{D(f(z))} = \frac{(1 - \gamma) z h'(z)}{D(f(z))} \frac{D(f(z))'}{D(f(z))} + \frac{1}{1 - \gamma} h(z) + z + \frac{1}{1 - \gamma} h'(z) \]
\[ = \frac{(1 - \gamma) h(z) + z + (1 - \gamma) h'(z)}{(1 - \gamma) h(z) + z}. \]

or
\[ 1 - \gamma \left\{ \frac{z(D(f(z))')'}{D(f(z))} - \gamma \right\} = h(z) + \frac{z h'(z)}{(1 - \gamma) h(z) + z}. \]

Therefore,
\[ \text{Re} \left\{ \frac{1}{1 - \gamma} \left( \frac{z(D(f(z))')'}{D(f(z))} - \gamma \right) \right\} = \text{Re} \left\{ h(z) + \frac{z h'(z)}{(1 - \gamma) h(z) + z} \right\}. \]

By using Lemma 2.1 for \( |z| = r < 1 \), we have
\[ \text{Re} \left\{ \frac{1}{1 - \gamma} \left( \frac{z(D(f(z))')'}{D(f(z))} - \gamma \right) \right\} \geq \text{Re} \left\{ h(z) + \frac{z h'(z)}{(1 - \gamma) h(z) + z} \right\}. \]

The right hand side of the above inequality (3.3) is positive for \( |z| < r_1 \), where \( r_1 \) is given by (3.2).

The sharpness can be seen by considering
\[ g(z) = \int_0^z \left( \frac{1 - (2\gamma - 1)r^\gamma}{1 - r^\gamma} \right) \, dt, \]
\[ h(z) = 1 - (2\gamma - 1)z^n \]
and
\[ f(z) = \int_0^z \left( \frac{1 - (2\gamma - 1)z^n}{1 - r^\gamma} \right)^2 \, dt. \]

The inclusion results for the classes \( S_\Lambda(\gamma, n) \), \( C_\Lambda(\gamma, n) \), \( K_\Lambda(\beta, \gamma) \) and \( C_\Lambda(\beta, \gamma) \), with \( n = 1 \), have been studied by Noor (1991).

We here deal with the converse case in general. \( \square \)

**Theorem 3.3.** Let \( f \in S(\gamma, n) \), \( \varpi \geq 0 \). Then \( f \in S_{\varpi + 1}(\gamma, n) \) for \( |z| < r_0(\varpi, \gamma) \) is given as
\[ r_0^\gamma = r_0^\gamma(\varpi, \gamma) = \frac{1 + \varpi}{(1 - \gamma + n) + \sqrt{(1 - \gamma + n)^2 - (1 + \varpi)(1 - 2\gamma + \varpi)}. \]

**Proof.** Since \( f \in S_{\varpi + 1}(\gamma, n) \), so we can write it as
\[ (z f(z))' = H(z) = (1 - \gamma) h(z) + \gamma, \]

where \( H \in P(\gamma, n) \) and so \( h \in P(0, n) = P_\Lambda \). Now using (1.7), we have
\[ D_{\varpi + 1}(f(z)) = \frac{1}{1 + \varpi} \left\{ (z f(z))' + \varpi \right\} = \frac{1}{1 + \varpi} \left\{ (1 - \gamma) h(z) + \gamma + \varpi \right\}. \]

Differentiating both sides logarithmically, we have
\[ \frac{z(D_{\varpi + 1}(f(z))')'}{D_{\varpi + 1}(f(z))} = \frac{(1 - \gamma) z h'(z)}{D_{\varpi + 1}(f(z))} + \frac{1}{1 - \gamma} h(z) + \gamma + \frac{1}{1 - \gamma} h'(z) \]

or
\[ 1 - \gamma \left\{ \frac{z(D_{\varpi + 1}(f(z))')'}{D_{\varpi + 1}(f(z))} - \gamma \right\} = h(z) + \frac{z h'(z)}{(1 - \gamma) h(z) + \gamma + \varpi}. \]

Therefore,
\[ \text{Re} \left\{ \frac{1}{1 - \gamma} \left( \frac{z(D_{\varpi + 1}(f(z))')'}{D_{\varpi + 1}(f(z))} - \gamma \right) \right\} = \text{Re} \left\{ h(z) + \frac{z h'(z)}{(1 - \gamma) h(z) + \gamma + \varpi} \right\}. \]

By using Lemma 2.1 for \( |z| = r < 1 \), we have
\[ \text{Re} \left\{ \frac{1}{1 - \gamma} \left( \frac{z(D_{\varpi + 1}(f(z))')'}{D_{\varpi + 1}(f(z))} - \gamma \right) \right\} \geq \text{Re} \left\{ h(z) + \frac{z h'(z)}{(1 - \gamma) h(z) + \gamma + \varpi} \right\}. \]
The right hand side of above inequality is positive if $r^\prime < r_0^\prime$ and so is given by (3.4).

As a special case, when $x = 0$, $n = 1$ and $\gamma = 0$, we obtain the radius of convexity $2 - \sqrt{3}$ for starlike functions. \[\square\]

**Theorem 3.4.** Let for $x \geq 0, f \in C_\gamma(x, n)$ in $E$. Then $f \in C_{x+1}(\gamma, n)$ for $|z| < r_0 = r_0(x, \gamma)$, where $r_0$ is given by (3.4).

**Proof.** By using the definition of $C_\gamma(x, n)$, we have

\[f \in C_\gamma(x, n) \iff D^tf \in C_\gamma(x, n) \text{ in } E.
\]

Also, since $z \in S(x, n)$ in $|z| < r_0$,

\[D^{x+1}(z^f) \in S(x, n) \text{ in } |z| < r_0.
\]

\[D^{x+1}(z^f) \in S(x, n) \text{ in } |z| < r_0.
\]

\[D^{x+1}(z^f) \in S(x, n) \text{ in } |z| < r_0.
\]

Which is the required result. \[\square\]

**Theorem 3.5.** Let for $x \geq 0, f \in K_\beta(\gamma, n)$ in $E$. Then $f \in K_{x+1}(\beta, \gamma, n)$ for $|z| < r_0 = r_0(x, \gamma)$, where $r_0$ is given by (3.4).

**Proof.** Since $f \in K_\beta(\gamma, n)$, there exists $g \in S_\gamma(n)$ such that

\[\frac{z(D^f(z))^\prime}{D^g(z)} = (1 - \beta)h(z) + \beta, \quad h \in P(0, n). \tag{3.5}\]

Also, since $g \in S_\gamma(n)$, we can write

\[\frac{z(D^g(z))^\prime}{D^g(z)} = (1 - \gamma)H(z) + \gamma, \quad H \in P(0, n). \tag{3.6}\]

Using (1.7), we have

\[\frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} = \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} + \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} + \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} + \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)}.
\]

By using (3.5) and (3.6), we have

\[\frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} = \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} + \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} + \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} + \frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)}.
\]

From (3.5), we have

\[z(D^f(z))^\prime = D^f(z)\{(1 - \beta)h(z) + \beta\}.
\]

Differentiating both sides, we have

\[z(D^f(z))^\prime = (1 - \beta)z^h(z)(D^f(z))^\prime + (D^f(z))^\prime\{(1 - \beta)h(z) + \beta\}.
\]

That is,

\[\frac{z(D^f(z))^\prime}{D^g(z)} = (1 - \beta)z^h(z) + \{(1 - \gamma)H(z) + \gamma\}\{(1 - \beta)h(z) + \beta\}. \tag{3.8}\]

Using (3.8) in (3.7), we obtain

\[
\left\{ \begin{array}{c}
1
\end{array} \right\}
= \left\{ \begin{array}{c}
\frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} - \beta
\end{array} \right\} - \beta = \left\{ \begin{array}{c}
\frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} - \beta
\end{array} \right\}.
\]

Now

\[
\left\{ \begin{array}{c}
1
\end{array} \right\} - \beta = \left\{ \begin{array}{c}
\frac{z(D^{x+1}(z^f))^\prime}{D^{x+1}(z^g)} - \beta
\end{array} \right\}.
\]

We note that right hand side is positive for $|z| < r_0 = r_0(x, \gamma)$ given by (3.4) and also $g \in S_\gamma(n)$ for $|z| < r_0$. Hence $f \in K_{x+1}(\beta, \gamma, n)$ for $|z| < r_0$. \[\square\]

As a special case, when $x = 0$, $\beta = 0$, $\gamma = 0$ and $n = 1$, we obtain the radius of quasi-convexity $2 - \sqrt{3}$ for close-to-convex functions.

Using the same method as in Theorem 3.4 with the relation (1.4), we can easily prove the following result.

**Theorem 3.6.** Let $f \in C_\gamma(x, \gamma) \text{ for } x \geq 0, \quad z \in E$. Then $f \in C_{x+1}(\gamma, n)$ for $|z| < r_0 = r_0(x, \gamma)$, where $r_0$ is given by (3.4).

**References**


