Some Inverse Solutions for Unsteady Fluid

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Abstract

Solutions for time dependent equations of an incompressible second-grade fluid are obtained using inverse method by assuming certain forms of the stream function.

Keywords and Phrases: Second-grade fluid, Exact solutions, Unsteady flow.
1. Introduction

The governing equations that describe the flow of a Newtonian fluid are the Navier-Stokes equations. These equations are non-linear partial differential equations and known exact solutions are few in number. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical and asymptotic methods. The equations of motion of non-Newtonian fluids are more complicated and non-linear than the Navier-Stokes equations. For this reason, there exists only a limited number of exact solutions. Due to the complexity of the equations, inverse methods described by Nemenyi [1] have become attractive. In these methods, solutions are found by assuming certain physical or geometrical properties of the flow field. Mohyuddin et al. [2, 5], Siddiqui et al. [3], Siddiqui [4], and Labropulu [6] used this method to study the flow problems of a second-grade fluid.

Unsteady flows of a second-grade fluid in a bounded region have been studied by Ting [7]. His work showed that the solution exists only if the coefficient of the higher order derivative in the governing equation is positive. Some unsteady unidirectional flows of second-grade fluids have been considered by Rajagopal [8], Hayat et al. [9 – 11] and Siddiqui et al. [12]. These works showed that the no-slip condition at the boundary for this type of flow suffices. A work in [13] showed that the solutions for unsteady flows of a second-grade fluid occupying the space above a plate are bounded if this coefficient is positive. However, this is not necessary for steady flows of second-grade fluids. The viscometric flows were studied by Markovitz and Coleman [14]. Lin and Tobak [15], Taylor [16] and Kovasznay [17] discussed some viscous flows for a chosen vorticity.

The paper is arranged in the following pattern: In section 2, governing equations and mathematical formulation of the problem is described. Section 3 consists of two parts. First part is the generalization of the Siddiqui’s work [4] for unsteady case and the second part is based on some special flows called as Riabouchinsky type flows and section 4 concludes some remarks. Stream functions, velocity fields and the pressure distributions are derived in each case. Moreover, the flow behaviour for streamlines are plotted in each case.
2. Governing Equations

The constitutive equation of an incompressible fluid of second-grade is [18]

\[ T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \]  

(2.1)

in which \( T \) is the Cauchy stress tensor, \(-pI\) denotes the indeterminate spherical stress and \( \mu, \alpha_1 \) and \( \alpha_2 \) are measurable material constants. They denote, respectively, the viscosity, elasticity and cross-viscosity. These material constants can be determined from viscometric flows for any real fluid. \( A_1 \) and \( A_2 \) are Rivlin-Ericksen tensors [18] and they denote, respectively, the rate of strain and acceleration.

\( A_1 = (\text{grad} V) + (\text{grad} V)^\top, \)  

(2.2)

\( A_2 = \frac{dA_1}{dt} + A_1 (\text{grad} V) + (\text{grad} V)^\top A_1. \)  

(2.3)

Here \( V \) is the velocity, \( \text{grad} \) the gradient operator, \( \top \) the transpose, and \( d/dt \) the material time derivative.

The basic equations governing the motion of a homogeneous incompressible second-grade neglecting the thermal effects, are the field equations

\[ \text{div} V = 0, \]  

(2.4)

\[ \rho \frac{dV}{dt} = \rho \chi + \text{div} T, \]  

(2.5)

where \( \rho \) is the density and \( \chi \) the body force.

Inserting (2.1) in (2.5) and making use of (2.2) and (2.3) we obtain the following vector equation

\[ \text{grad} \left[ \frac{1}{2} \rho |V|^2 + p - \alpha_1 \left( V \cdot \nabla^2 V + \frac{1}{4} |A_1|^2 \right) \right] + \rho [V_t - V \times (\nabla \times V)] \]

\[ = \mu \nabla^2 V + \alpha_1 \left[ \nabla^2 V_t + \nabla^2 (\nabla \times V) \times V \right] + (\alpha_1 + \alpha_2) \text{div} A_1^2 + \rho \chi \]  

(2.6)

in which \( \nabla^2 \) is the Laplacian operator, \( V_t = \partial V / \partial t \), and \(|A_1|\) is the usual norm of matrix \( A_1 \). If this model is required to be compatible with thermodynamics, then the material constants must meet the restrictions [19, 20]

\[ \mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \]  

(2.7)
On the other hand, experimental results of tested fluids of second-grade showed that \( \alpha_1 < 0 \) and \( \alpha_1 + \alpha_2 \neq 0 \) which contradicts the above conditions and imply that such fluids are unstable. This controversy is discussed in detail in [21]. However, in this paper we will discuss both cases, \( \alpha_1 \geq 0 \) and \( \alpha_1 < 0 \).

Let us consider the motion of an unsteady incompressible second-grade fluid in which the velocity field is of the form

\[
\mathbf{V}(x, y, z, t) = [u(x, y, t), v(x, y, t), 0],
\]

(2.8)

where \( u \) and \( v \) are the velocity components in \( x \)– and \( y \)–directions respectively.

Inserting (2.8) in (2.4) and (2.6) and making use of the assumption (2.7) we obtain, in the absence of body forces, the following equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

(2.9)

\[
\frac{\partial \hat{p}}{\partial x} + \rho \left[ \frac{\partial u}{\partial t} - v \omega \right] = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 u - \alpha_1 v \nabla^2 \omega,
\]

(2.10)

\[
\frac{\partial \hat{p}}{\partial y} + \rho \left[ \frac{\partial v}{\partial t} + u \omega \right] = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 v + \alpha_1 u \nabla^2 \omega,
\]

(2.11)

where

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},
\]

(2.12a)

\[
\hat{p} = \rho + \frac{1}{2} \rho \left( u^2 + v^2 \right) - \alpha_1 \left[ u \nabla^2 u + v \nabla^2 v + \frac{1}{4} |A_1|^2 \right],
\]

(2.12b)

\[
|A_1|^2 = 4 \left( \frac{\partial u}{\partial x} \right)^2 + 4 \left( \frac{\partial v}{\partial y} \right)^2 + 2 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2.
\]

(2.12c)

On setting \( \alpha_1 = 0 \) in (2.10) and (2.11) and considering only steady case we recover the equations for Newtonian fluid [22].

Equations (2.9) – (2.11) are three partial differential equations for three unknown functions \( u, v \) and \( \hat{p} \) of the variables \( (x, y, t) \). Once the velocity field is determined, the pressure field (2.12b) can be calculated by integrating (2.10) and (2.11). Note that the equation for the vertical component \( w \) is identically zero.
Eliminating pressure in (2.10) and (2.11), by applying the integrability condition \( \partial^2 \hat{p}/\partial x \partial y = \partial^2 \hat{p}/\partial y \partial x \), we get the vorticity equation

\[
\rho \left[ \frac{\partial \omega}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \omega \right] = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^2 \omega + \alpha_1 \left[ \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \nabla^2 \omega \right].
\] (2.13)

Introduce the flow function of the following form:

\[
u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x};
\] (2.14)

where \( \psi (x, y, t) \) is the stream function, called the Stokes stream function. We see that the continuity equation (2.9) is satisfied identically and (2.14) in (2.13) yields the following compatibility equation

\[
\rho \left[ \frac{\partial}{\partial t} \nabla^2 \psi - \left\{ \psi, \nabla^2 \psi \right\} \right] = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \nabla^4 \psi - \alpha_1 \left\{ \psi, \nabla^4 \psi \right\}
\] (2.15)

in which

\[
\nabla^4 = \nabla^2 \cdot \nabla^2,
\] (2.16)

and the vorticity vector and Poisson bracket in terms of the stream function are respectively given by

\[
\omega = -\nabla^2 \psi,
\] (2.17)

\[
\left\{ \psi, \nabla^2 \psi \right\} = \frac{\partial \psi}{\partial x} \frac{\partial (\nabla^2 \psi)}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial (\nabla^2 \psi)}{\partial x}.
\] (2.18)

### 3. Solutions of Some Special Types

#### 3.1. Solution of the Type \( \psi (x, y, t) = \xi (x, t) + \eta (y, t) \)

We consider the plane unsteady flow and examine the solution of (2.15) of the form

\[
\psi (x, y, t) = \xi (x, t) + \eta (y, t),
\] (3.1.1)

where \( \xi \) and \( \eta \) are arbitrary functions of the variables \( x, t \) and \( y, t \) respectively. Substituting (3.1.1) in (2.15) we obtain the following equation

\[
\rho \left[ \xi_{xxx} (x, t) + \eta_{yyy} (y, t) + \eta_y (y, t) \xi_{xxx} (x, t) - \xi_x (x, t) \eta_{yyy} (y, t) \right]
\]
in which the subscripts indicates the derivatives with respect to the variables \( x, y \) and time \( t \).

Let us consider a particular solution of (3.1.2) of the form

\[
\xi (x, t) = -V_1 + \phi (s_1), \quad s_1 = x + V_1 t,
\]

\[
\eta (y, t) = -V_2 + \theta (s_2), \quad s_2 = y + V_2 t,
\]

where \( V_1 \) and \( V_2 \) are constants and \( \phi \) and \( \theta \) satisfy the differential equation

\[
\rho \left[ V_1 \phi''' (s_1) + V_2 \theta''' (s_2) + \theta' (s_2) \phi''' (s_1) - \phi' (s_1) \theta''' (s_2) \right] = \mu \left[ \phi^V (s_1) + V_2 \theta^V (s_2) \right],
\]

in which primes (\( ' \)), \( IV \) and \( V \) in the superscript indicates the fourth and fifth derivatives with respect to its arguments. We see that the above equation is highly non-linear and non-separable and its solution in the present form is not easy to obtain. In order to find its solution we assume the following [4]

\[
\phi (s_1) = A s_1 + B e^{a s_1},
\]

\[
\theta (s_2) = C s_2 + D e^{b s_2}
\]

and obtain the following equation

\[
\rho \left[ (V_1 + C) a^3 B e^{a s_1} + (V_2 - A) b^3 D e^{b s_2} + \frac{abBD (a^2 - b^2)}{e^{a s_1 + b s_2}} \right] = \mu \left[ a^4 B e^{a s_1} + b^4 D e^{b s_2} + (V_1 + C) a^5 B e^{a s_1} + (V_2 - A) b^5 D e^{b s_2} + \frac{abBD (a^4 - b^4)}{e^{a s_1 + b s_2}} \right],
\]

where \( A, B, C, D, a \) and \( b \) are arbitrary constants.

The following three equations are obtained from (3.1.8)

\[
\rho (V_1 + C) = \mu a + \alpha_1 a^2 (V_1 + C),
\]

\[
\rho (V_2 - A) = \mu b^2 + \alpha_1 b^2 (V_2 - A),
\]
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\[(b^2 - a^2) \left[ \rho - \alpha_1 \left( a^2 + b^2 \right) \right] = 0. \quad (3.1.11)\]

From (3.1.9) and (3.1.10) we easily obtain the values of \(A\) and \(C\) i.e.

\[A = -\frac{\mu b}{\rho - \alpha_1 b^2} + V_2, \quad (3.1.12)\]

\[C = \frac{\mu a}{\rho - \alpha_1 a^2} - V_1, \quad (3.1.13)\]

and (3.1.11) is satisfied if either

\[b^2 - a^2 = 0 \quad (3.1.14)\]

or

\[\rho = \alpha_1 \left( a^2 + b^2 \right). \quad (3.1.15)\]

We have three different cases which we discuss one by one:

**Case 1.** \(b = a, \rho \neq \alpha_1 \left( a^2 + b^2 \right)\)

Then the stream function given by (3.1.1), after using (3.1.3), (3.1.4), (3.1.6), (3.1.7), (3.1.12), and (3.1.13) becomes

\[\psi (x, y, t) = -V_1 (1 + y) - V_2 (1 - x) + Be^{a(x+V_1 t)} + De^{a(y+V_2 t)} + \frac{\mu a}{\rho - \alpha_1 a^2} [(y - x) + (V_2 - V_1) t] \quad (3.1.16)\]

and from (2.14), the velocity components have the form

\[u = -V_1 + \frac{\mu a}{\rho - \alpha_1 a^2} + Da e^{a(y+V_2 t)}, \quad (3.1.17)\]

\[v = -V_2 + \frac{\mu a}{\rho - \alpha_1 a^2} - Ba e^{a(x+V_1 t)}. \quad (3.1.18)\]

In order to find the pressure field (2.12b) we substitute the velocity components (3.1.17) and (3.1.18) in (2.10) and (2.11) and then integrate the resulting equations to obtain

\[p = p_0 (t) - \frac{1}{2} \rho \left( \bar{a}^2 + \bar{b}_1^2 \right) - \mu B a^3 y e^{a(x+V_1 t)} + \left( \rho - \alpha_1 a^2 \right) \left[ a^2 B (\bar{a} + V_1) y e^{a(x+V_1 t)} + a^2 D B e^{a(x+V_1 t)+a(y+V_2 t)} + \bar{b}_1 B a e^{a(x+V_1 t)} - \frac{1}{2} B^2 a^2 e^{2a(x+V_1 t)} \right] \]

where
\[ \alpha = -V_1 + \frac{\mu a}{\rho - \alpha_1 a^2}, \quad \beta_1 = -V_2 + \frac{\mu a}{\rho - \alpha_1 a^2}, \]
where \( p_0 (t) \) is an arbitrary function of \( t \), known as the reference pressure.

The functional form of streamline for \( \psi = \Omega_1 (\Omega_1 \text{ is a constant}) \) is given by
\[ y = \frac{1}{V_1 - \varepsilon} \left[ B e^{a(x + V_1 t)} - \{V_1 + (1 - x) V_2\} - \{(V_1 - V_2) t + x\} \varepsilon - \Omega_1 \right] \]
\[ - \frac{1}{a} \text{ProductLog} \left[ \frac{-D a}{V_1 - \varepsilon} \exp \left[ \frac{-a}{V_1 - \varepsilon} \left\{ -B e^{a(x + V_1 t)} + V_1 + (1 - x) V_2 \right\} \right] \right], \]
where
\[ \varepsilon = \frac{\nu a}{1 - \Lambda a^2}, \quad 1 - \Lambda a^2 \neq 0, \]
\( \nu = \mu/\rho \) is the kinematic viscosity and \( \Lambda = \alpha_1/\rho \) is the second-grade elastic parameter.

Streamline are shown in Fig. 1. for \( B = D = V_1 = V_2 = a = 1, \mu/\rho = 0.5, \]
\( t = 1, \alpha_1/\rho = 0.1, \psi = 15, 20, 25, 30, 40. \)

**Case 2.** \( b = -a, \rho \neq \alpha_1 (a^2 + b^2) \)
The expressions for \( \psi, u, v, \) and \( p \) are
\[ \psi (x, y, t) = -V_1 (1 + y) - V_2 (1 - x) + B e^{a(x + V_1 t)} \]
\[ + D e^{-a(y + V_2 t)} + \frac{\mu a}{\rho - \alpha_1 a^2} [(y + x) + (V_2 + V_1) t], \]
\[ u = -V_1 + \frac{\mu a}{\rho - \alpha_1 a^2} - D a e^{-a(y + V_2 t)}, \]
\[ v = -V_2 - \frac{\mu a}{\rho - \alpha_1 a^2} - B a e^{a(x + V_1 t)}, \]
\[ p = p_0 (t) - \frac{1}{2} \mu \rho \left( \bar{a}^2 + \bar{b}^2 \right) - \mu B a^3 y e^{a(x + V_1 t)} \]
\[ + (\rho - \alpha_1 a^2) \left[ a^2 B (\bar{a} + V_1) y e^{a(x + V_1 t)} + a^2 B e^{a(x + V_1 t) - a(y + V_2 t)} \right] 
\[ + \bar{b}^2 B a e^{a(x + V_1 t)} - \frac{1}{2} B^2 a^2 e^{2a(x + V_1 t)} \]
\[ + \alpha_1 \left[ D^2 a^4 e^{-2a(y + V_2 t)} + B^2 a^4 e^{2a(x + V_1 t)} \right]. \]
where
\[ a = -V_1 + \frac{\mu a}{\rho - \alpha_1 a^2}, \quad b_2 = -V_2 - \frac{\mu a}{\rho - \alpha_1 a^2}, \]
and the functional form of streamline for \( \psi = \Omega_2 \) (\( \Omega_2 \) is a constant) is given as
\[
y = \frac{1}{V_1 - \varepsilon} \left[ B e^{a(x+V_1 t)} - \{ V_1 + (1 - x) V_2 \} + \{(V_1 + V_2) t + x \} \varepsilon - \Omega_2 \right] \\
+ \frac{1}{a} \text{ProductLog} \left[ \frac{Da}{V_1 - \varepsilon} \exp \left[ \frac{-a}{V_1 - \varepsilon} \left\{ \frac{B e^{a(x+V_1 t)} - V_1 - (1 - x) V_2}{V_1 + (1 - x) V_2} \right\} \right] \right].
\]

Streamlines are shown in Fig. 2 for \( B = D = V_1 = V_2 = a = 1, \mu/\rho = 0.5, t = 0.1, \alpha_1/\rho = 0.1, \psi = 15, 20, 25, 30, 40. \)

**Case 3.** \( b^2 - a^2 \neq 0 \)
Then we must have
\[ \rho = \alpha_1 (a^2 + b^2) \]
and the expressions for \( \psi, u, v, \) and \( p \) are of the following form
\[
\psi(x, y, t) = -V_1 (1 + y) - V_2 (1 - x) + \frac{\mu b}{\rho - \alpha_1 b^2} (x + V_1 t) \\
+ \frac{\mu a}{\rho - \alpha_1 a^2} (y + V_2 t) + B e^{a(x+V_1 t)} + D e^{b(y+V_2 t)},
\]
\[
u = -V_2 - \frac{\mu b}{\rho - \alpha_1 b^2} - B a e^{a(x+V_1 t)},
\]
\[
p = p_0 (t) - \frac{1}{2} \mu \left( a^2 + b^2 \right) - \frac{1}{2} \rho (a^2 + b^2) - B e^{a(x+V_1 t)} \]
\[
+ (\rho - \alpha_1 a^2) \left[ \frac{a^2 B (\pi + V_1) ye^{a(x+V_1 t)}}{2} + \frac{a^2 D B e^{a(x+V_1 t)+b(y+V_2 t)}}{2} \right] \\
+ \alpha_1 \left[ \frac{B^2 a^2 e^{2a(x+V_1 t)}}{2} + \frac{D^2 b^2 e^{2b(y+V_2 t)}}{2} \right],
\]
where
\[ \bar{a} = -V_1 + \frac{\mu a}{\rho - \alpha_1 a^2}, \quad \bar{b} = -V_2 + \frac{\mu b}{\rho - \alpha_1 b^2}, \]
whereas the functional form in this case for $\psi = \Omega_3$ ($\Omega_3$ is constant) is

$$
y = \frac{1}{V_1 - \varepsilon} \left[ B e^{a(x + V_1 t)} - V_1 + (x - 1 + t \varepsilon) V_2 - V_1 V_2 t - x \varepsilon_1 - \Omega_3 \right]
$$

$$
- \frac{1}{b} \text{ProductLog} \left[ \frac{-bD}{V_1 - \varepsilon} \exp \left[ \frac{b}{V_1 - \varepsilon} \left\{ \frac{B e^{a(x + V_1 t)} - V_1 - V_2 - \Omega_3}{V_2 (x + V_1 t) - \varepsilon_1 (x + V_1 t) \varepsilon} \right\} \right] \right],
$$

(3.1.30)

where

$$
\varepsilon_1 = \frac{\nu b}{1 - \Lambda b^2}, \quad 1 - \Lambda b^2 \neq 0.
$$

Streamlines for $B = D = V_1 = V_2 = a = 1$, $b = 3$, $t = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $\psi = 15, 20, 25, 30, 40$, is depicted in Fig. 3.

The alternate forms of (3.1.26) may be written as

$$
\psi (x, y, t) = -V_1 (1 + y) - V_2 (1 - x) + \frac{\mu b}{\rho - \alpha_1 b^2} (x + V_1 t)
$$

$$
+ \frac{\mu a}{\alpha_1 b^2} (y + V_2 t) + B e^{a(x + V_1 t)} + D e^{b(y + V_2 t)},
$$

(3.1.31)

$$
\psi (x, y, t) = -V_1 (1 + y) - V_2 (1 - x) + \frac{\mu b}{\rho - \alpha_1 a^2} (x + V_1 t)
$$

$$
+ \frac{\mu a}{\rho - \alpha_1 a^2} (y + V_2 t) + B e^{a(x + V_1 t)} + D e^{b(y + V_2 t)}.
$$

(3.1.32)

**Remark 3.1** The solution (3.1.16) with $V_1 = V_2 = 0$ and $\alpha_1 = 0$ gives the Berker’s solution [22] and the Siddiqui’s solutions [4] can be recovered by taking $V_1 = V_2 = 0$. 
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Fig. 1.

Fig. 2.
We now consider Riabouchinsky type flows in order to solve (2.15).

3.2. Solution of the Type $\psi(x, y, t) = y\xi(x, t)$

In order to obtain another class of solution of (2.15) we substitute

$$\psi(x, y, t) = y\xi(x, t) \quad (3.2.1)$$

into (2.15) and get the following equation

$$\rho \left[ \xi'' - \xi'\xi'' + \xi\xi''' \right] = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \xi^{IV} - \alpha_1 \left[ \xi'\xi^{IV} - \xi^{V} \right], \quad (3.2.2)$$

where $\xi(x, t)$ is an arbitrary function of $x, t$, subscript is the derivative with respect to time and primes denote the derivative with respect to $x$.

The first integral of (3.2.2) is given by

$$\rho \left[ \xi'_t - (\xi')^2 + \xi\xi'' \right] = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \xi'' - \alpha_1 \left[ \xi\xi^{IV} - 2\xi'\xi''' - (\xi'')^2 \right] + f(t), \quad (3.2.3)$$

where $f(t)$ is an arbitrary function of $t$. When this function is taken to be zero there is a particular solution of type
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\[ \xi(x, t) = -V + F(x + Vt) = -V + F(s), \quad (3.2.4) \]

where \( V \) is a constant and \( F \) satisfies the differential equation

\[
\rho \left[ F \frac{d^2 F}{ds^2} - \left( \frac{dF}{ds} \right)^2 \right] = \mu \frac{d^3 F}{ds^3} + \alpha_1 \left[ F \frac{d^4 F}{ds^4} - 2 \frac{dF}{ds} \frac{d^3 F}{ds^3} + \left( \frac{d^2 F}{ds^2} \right)^2 \right]. \quad (3.2.5)
\]

A particular solution of (3.2.5) can be obtained by assuming

\[ F(s) = U \left( 1 + c e^{\lambda s} \right), \quad (3.2.6) \]

in which \( U, c \) and \( \lambda \) are arbitrary real constants. Making use of (3.2.6) into (3.2.5) we have

\[ U = \frac{\mu \lambda}{\rho - \alpha_1 \lambda^2}, \quad (3.2.7) \]

and thus from (3.2.1)

\[ \psi(x, y, t) = -Vy + \frac{\mu \lambda}{\rho - \alpha_1 \lambda^2} \left[ 1 + ce^{\lambda(x+Vt)} \right] y. \quad (3.2.8) \]

The velocity components (2.14) and the pressure field (2.12b) become

\[ u = -V + \frac{\mu \lambda}{\rho - \alpha_1 \lambda^2} \left[ 1 + ce^{\lambda(x+Vt)} \right], \quad (3.2.9) \]

\[ v = -\frac{\mu \lambda}{\rho - \alpha_1 \lambda^2} c e^{\lambda(x+Vt)}, \quad (3.2.10) \]

\[ p = p_0(t) - \frac{1}{2} \rho \overline{a}_2^2 - \mu \overline{\lambda}_1 \lambda^2 ye^{\lambda(x+Vt)} \]

\[ + (\rho - \alpha_1 \lambda^2) \left[ \left( \overline{\lambda}_1 \lambda y (V + \overline{a}_2) - \overline{a}_2 \lambda \right) e^{\lambda(x+Vt)} \right. \]

\[ + \left. \left( \overline{\lambda}_1 \lambda y - \frac{1}{2} \left( \overline{\lambda}_1 + \lambda^2 \right) \right) e^{2\lambda(x+Vt)} \right] \]

\[ + \alpha_1 \lambda^2 \left[ \overline{\lambda}_1 + \frac{3}{2} \lambda^2 \right] e^{2\lambda(x+Vt)}, \quad (3.2.11) \]

where

\[ \overline{a}_2 = -V + \frac{\mu \lambda}{\rho - \alpha_1 \lambda^2} = -V + \overline{\lambda}, \quad \overline{\lambda}_1 = \overline{\lambda} \lambda c. \]
The streamline flow for $\psi = \Omega_4$ is given by the functional form

$$y = \frac{-\Omega_4}{V - \varepsilon_2 \{1 + c e^{\lambda(x+Vt)}\}}, \quad (3.2.12)$$

where

$$\varepsilon_2 = \frac{\nu \lambda}{1 - \Lambda \lambda^2}, \quad 1 - \Lambda \lambda^2 \neq 0.$$  

Fig. 4. shows the streamlines for $V = 1$, $\lambda = 2$, $t = 0.5$, $c = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $\psi = 15, 20, 25, 30, 40$ and in Fig. 7. streamlines are traced for $t = .1, .5, 1, 1.5, 2$, $\psi = 15$, $V = 1$, $\lambda = 2$, $c = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$.

### 3.3. Solutions of the Type $\psi (x, y, t) = y\xi (x, t) + \eta (x, t)$

Inserting

$$\psi (x, y, t) = y\xi (x, t) + \eta (x, t) \quad (3.3.1)$$
in (2.15) we obtain the following differential equations satisfied by $\xi$ and $\eta$

\begin{align*}
\rho [\xi'' + \xi'\xi'' + \xi \xi'''] &= \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \xi^{IV} + \alpha_1 \left[\xi \xi^{IV} - \xi' \xi^{IV'}\right], \quad (3.3.2) \\
\rho [\eta'' - \eta'\xi'' + \xi \eta'''] &= \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \eta^{IV} + \alpha_1 \left[\xi \eta^{IV} - \eta' \xi^{IV'}\right]. \quad (3.3.3)
\end{align*}

The first integral of the above equations is

\begin{align*}
\rho \left[\xi_t - (\xi')^2 + \xi \xi''\right] &= \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \xi'' + \alpha_1 \left[\xi \xi^{IV} - 2\xi' \xi''' - (\xi'')^2\right] + f_1 (t), \quad (3.3.4) \\
\rho [\eta_t - \xi \eta'' - \eta' \xi'] &= \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \eta''' + \alpha_1 \left[\xi \eta^{IV} - \xi' \eta''' - \eta' \xi''' + \eta'' \xi''\right] + f_2 (t), \quad (3.3.5)
\end{align*}

where $f_1 (t)$ and $f_2 (t)$ are arbitrary functions of $t$. On taking $f_1 (t) = f_2 (t) = 0$ we observe that the differential equation for $\xi$ is the same as in section 3.2. When $\xi$ is known, the second equation is a linear partial differential equation for the determination of $\eta (x, t)$. A particular solution of this equation is seen to be $\eta = \xi (x, t)$ and this fact is useful for the purpose of obtaining further solutions. In particular, if

$$\xi (x, t) = -V + F (x + Vt) = -V + F (s), \quad (3.3.6)$$
Some Inverse Solutions for Unsteady Fluid

\[ \eta(x, t) = -V + G(x + V t) = -V + G(s), \]  
(3.3.7)

we have the following differential equation for the function \(G\)

\[ \rho \left[ F \frac{d^2 G}{ds^2} - \frac{dF}{ds} \frac{dG}{ds} \right] = \mu \frac{d^2 G}{ds^2} + \alpha_1 \left[ F \frac{d^4 G}{ds^4} - \frac{dF}{ds} \frac{d^4 G}{ds^4} - \frac{dG}{ds} \frac{dF}{ds} + \frac{d^2 G}{ds^2} \frac{d^2 F}{ds^2} \right]. \]  
(3.3.8)

A particular solution is known to be \(G = F(s)\). Inserting the solution (3.2.6) in (3.3.8) and simplifying we obtain

\[ \alpha_1 \left( 1 + ce^{\lambda s} \right) \frac{d^3 \Gamma}{ds^3} + \left( \mu / U - \alpha_1 c \lambda e^{\lambda s} \right) \frac{d^2 \Gamma}{ds^2} \]
\[ = + \left[ \rho - \left( \rho - \alpha_1 \lambda^2 \right) ce^{\lambda s} \right] \frac{d\Gamma}{ds} + c \lambda \left( \rho - \alpha_1 \lambda^2 \right) \Gamma \]  
(3.3.9)

where \(\Gamma = dG/ds\).

Consequently, we may reduce the order of the equation by means of the consecutive substitutions \(\Gamma(s) = \theta(s) e^{\lambda s}\) and \(\theta'(s) = R(s)\) to obtain

\[ \alpha_1 \left( 1 + ce^{\lambda s} \right) \frac{d^2 R}{ds^2} + \left[ 2 \alpha_1 \lambda \left( 1 + ce^{\lambda s} \right) + \rho / \lambda \right] \frac{dR}{ds} - \left[ \left( \rho + \alpha_1 \lambda^2 \right) + \left( 2 \alpha_1 \lambda^2 - \rho \right) e^{\lambda s} \right] R = 0. \]  
(3.3.10)

The solution of (3.3.10) for \(\lambda = 1\) and \(c = 0\) is given by

\[ R(s) = C_5 e^{-s} + C_6 e^{-[(\alpha_1 + \rho)/\alpha_1]s}. \]  
(3.3.11)

The backward substitution gives the value of \(\eta(x, t)\)

\[ \eta(x, t) = -V - C_5 (x + V t) + \frac{\alpha_1^2}{\rho (\alpha_1 + \rho)} C_6 e^{-\left( \rho/\alpha_1 \right)(x + V t)} + C_7 e^{(x + V t)} + C_8, \]  
(3.3.12)

where \(C_r\) \((r = 5, 6, 7, 8)\) are arbitrary constants. The stream function, the velocity components and the pressure field in this case are respectively given as

\[ \psi(x, y, t) = -V + \left( \frac{\mu}{\rho - \alpha_1} - V \right) y - C_5 (x + V t) \]
\[ + \frac{\alpha_1^2}{\rho (\alpha_1 + \rho)} C_6 e^{-\left( \rho/\alpha_1 \right)(x + V t)} + C_7 e^{(x + V t)} + C_8, \]  
(3.3.13)
\[ u = -V + \frac{\mu}{\rho - \alpha_1}, \quad (3.3.14) \]

\[ v = C_5 + \frac{\alpha_1}{(\alpha_1 + \rho)} C_6 e^{-\left(\rho/\alpha_1\right)(x+Vt)} - C_7 e^{(x+Vt)}, \quad (3.3.15) \]

\[ p = p_0(t) - \frac{1}{2} \rho \left[ \frac{a_2^2 + C_7^2}{(1-\rho/\alpha_1)} C_6 e^{-2\left(\rho/\alpha_1\right)(x+Vt)} + 2 \left(\frac{1-\rho/\alpha_1}{\alpha_1 - \rho}\right) C_7 e^{(x+Vt)} \right. \]

\[ + \left. \alpha_1 \left[ C_7^2 e^{2(x+Vt)} + \frac{\rho^2 \pi^2}{\alpha_1^4} e^{2\left(\rho/\alpha_1\right)(x+Vt)} - C_7^2 \frac{\rho^2 \pi}{\alpha_1^4} e^{(x+Vt)} \right] \right], \quad (3.3.16) \]

where

\[ \alpha = \frac{\alpha_1}{\alpha_1 + \rho}, \quad \alpha_2 = -V + \frac{\mu}{\rho - \alpha_1}, \]

and the streamline for \( \psi = \Omega_5 \) (\( \Omega_5 \) is constant) is given by the functional form

\[ y = \frac{\Lambda - 1}{V (\Lambda - 1) + \nu} \]

\[ \times \left[ -\Omega_5 - C_5 (x + Vt) + \frac{\Lambda^2}{1 + \Lambda} C_6 e^{-\left(1/\Lambda\right)(x+Vt)} + C_7 e^{(x+Vt)} + C_8 \right]. \quad (3.3.17) \]

Streamlines are given in Fig. 5 for \( \lambda = 1, \ V = 0, \ t = 0.1, \ \mu/\rho = 0.5, \ \alpha_1/\rho = 0.1, \ C_5 = C_6 = C_7 = C_8 = 1, \ \psi = 15, 20, 25, 30, 40. \]
Fig. 5.

Fig. 6.
4. Concluding Remarks

In this paper, the analytical solutions of non-linear equations governing the flow of an unsteady second-grade fluid are obtained by assuming certain forms of the stream function. The expressions for velocity profile, streamline and pressure distribution are constructed in each case. Our results indicate that velocity, stream function and pressure are strongly dependent upon the material parameter $\alpha_1$ of the second-grade fluid. It is shown through graphs that increase in $\alpha_1$ leads to decrease in velocity and decrease in $\alpha_1$ leads to increase in velocity (see Fig. 6). Also, the present analysis is more general and several results of various authors (as already mentioned in the text) can be recovered in the limiting cases.
References


